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MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

STEPHEN A. JENNINGS, *EDITOR*

EDITORIAL CORRESPONDENCE should be sent to the EDITOR-ELECT, G. N. WOLLAN, Department of Mathematics, Purdue University, Lafayette, Indiana 47907. Articles should be typewritten and triple-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

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A PROBLEM IN GEOMETRICAL PROBABILITY

ERIC LANGFORD, Naval Postgraduate School

This paper is dedicated to Professor W. Randolph Church, 1904–1969.

Introduction. In 1955, Frank Hawthorne proposed the following problem in the *American Mathematical Monthly* [1]: *If three points are selected at random in a rectangle $A \times 2A$, what is the probability that the triangle so determined is obtuse?*

Although the problem does not at first glance seem to be especially difficult, no solutions were received. In 1962, C. S. Ogilvy, in his book *Tomorrow's Math* [2], mentioned Hawthorne's problem and noted that it was still at that time unsolved. To this day, the editors of the MONTHLY have failed to receive any solutions.

The following more general problem is immediately suggested: *Let there be given three points at random in an arbitrary rectangle. What is the probability that the triangle thus formed is obtuse?*

This more general problem was first posed to me several years ago by Professor Roger Pinkham; Professor Pinkham has informed me that he obtained a partial solution to it (including a solution to Hawthorne's problem) around 1956, but that the solution was never published.

In this paper we shall present a complete solution to the obtuse triangle problem for an arbitrary rectangle.

Analysis. Let $P(L)$ be the probability that three points chosen at random in a rectangle with dimensions $1 \times L$ form an obtuse triangle. The desired probability is invariant under change of scale; thus once we have determined $P(L)$, we have solved the general problem.

To state the problem precisely, let R denote the rectangle $\{(x, y): 0 \leqq x \leqq 1$ and $0 \leqq y \leqq L\}$ and let X_1, X_2, X_3, Y_1, Y_2, and Y_3 be independent random variables where each X_i is distributed uniformly on the unit interval $[0, 1]$ and where each Y_i is distributed uniformly on the interval $[0, L]$. Consider the following three random points in R which they determine: $P_1 = (X_1, Y_1)$, $P_2 = (X_2, Y_2)$, and $P_3 = (X_3, Y_3)$. With probability one, these points form a nondegenerate triangle $P_1P_2P_3$. Defining $P(L)$ as above to be the probability that triangle $P_1P_2P_3$ is obtuse, we see that $P(L) = Pr\{\text{Angle } P_1 \text{ is obtuse}\} + Pr\{\text{Angle } P_2 \text{ is obtuse}\} + Pr\{\text{Angle } P_3 \text{ is obtuse}\}$, since a triangle can have at most one obtuse angle. By symmetry, it follows that

$$P(L) = 3Pr\{\text{Angle } P_1 \text{ is obtuse}\}.$$

If we let P_1P_2 be the vector from P_1 to P_2 and P_1P_3 be the vector from P_1 to P_3, then

$$\cos P_1 = \frac{(P_1P_2)\cdot(P_1P_3)}{|P_1P_2|\ |P_1P_3|}.$$

Now angle P_1 is obtuse iff $\cos P_1$ is negative, and evidently $\cos P_1$ is negative iff the dot product

$$(P_1P_2)\cdot(P_1P_3) = (X_2 - X_1)(X_3 - X_1) + (Y_2 - Y_1)(Y_3 - Y_1)$$

is negative. If we define new random variables

$$X = (X_2 - X_1)(X_3 - X_1) \quad \text{and} \quad Y = (Y_2 - Y_1)(Y_3 - Y_1),$$

then we have

$$P(L) = 3Pr\{X + Y < 0\}.$$

Let $F(x)$ denote the cumulative distribution function (CDF) of X; then by a simple change of scale, the CDF of Y will be given by $G(y) = F(y/L^2)$; thus it is sufficient to determine $F(x)$. Once this is done, we see that X and Y are independent so that $P(L)$ can be expressed as the following Riemann-Stieltjes integral:

$$P(L) = 3\int_{-\infty}^{\infty} F(-x/L^2)dF(x). \tag{1}$$

(We note that for all values of L, the range of integration is actually finite.)

Computation of $F(x)$. We shall compute first the conditional CDF $F_1(x, a) = Pr\{X \leqq x \text{ given that } X_1 = a\} = Pr\{(X_2 - a)(X_3 - a) \leqq x\}$. Evidently we need only consider $a \in [0, 1]$; $F(x)$ is then found by using the relationship

$$F(x) = \int_0^1 F_1(x, a)da.$$

We examine the case $x > 0$ first. Consider the unit square in the $X_2 - X_3$ plane. For fixed $x > 0$, the graph of the equation $(X_2 - a)(X_3 - a) = x$ is a hyperbola with asymptotes $X_2 = a$ and $X_3 = a$ as shown in Figure 1. The region between the two branches of the hyperbola will determine the set where $(X_2 - a)(X_3 - a) \leqq x$; since the joint distribution of X_2 and X_3 is uniform over the square, $F_1(x, a)$ is simply the shaded area in Figure 1.

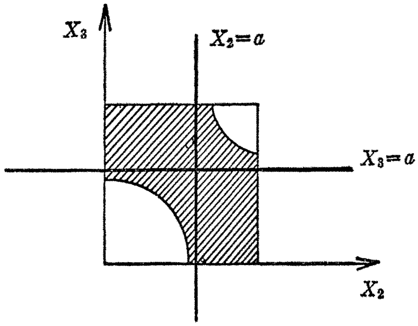

FIG. 1. The shaded region is $\{(X_2, X_3) : (X_2 - a)(X_3 - a) \leqq x\}$ when x is positive.

Figure 1 is apt to be misleading, for it implies that there are always two unshaded areas: A_1 in the lower left corner and A_2 in the upper right corner.

This is not true in general, since depending on the relative magnitudes of x and a, one or both of these regions can disappear. The intersection of the lower branch of the hyperbola with either the X_2- or X_3-axis is seen to be at $a-(x/a)$, so that for the region A_1 to be present we need that $a-(x/a)\geqq 0$, i.e., that $a^2\geqq x$. By a similar argument it is seen that we must have $(1-a)^2\geqq x$ in order for the region A_2 to be present.

Let us define $A_1(x, a)$ to be the area of A_1 if A_1 is present, and 0 otherwise; thus if $a^2\geqq x$, we have that

$$\begin{aligned} A_1(x, a) &= \int_0^{a-x/a}\left(a+\frac{x}{X_3-a}\right)dX_3 \\ &= x\log x - 2x\log a + a^2 - x. \end{aligned} \tag{2}$$

Defining $A_2(x, a)$ analogously, we see that if $(1-a)^2\geqq x$, then

$$\begin{aligned} A_2(x, a) &= \int_{a+x/(1-a)}^{1}\left(1-a-\frac{x}{X_3-a}\right)dX_3 \\ &= x\log x - 2x\log(1-a) + (1-a)^2 - x. \end{aligned} \tag{3}$$

With this notation, evidently,

$$F_1(x, a) = 1 - A_1(x, a) - A_2(x, a),$$

so that for $x>0$,

$$F(x) = 1 - \int_0^1 A_1(x, a)da - \int_0^1 A_2(x, a)da.$$

Certainly $F(x)=1$ if $x\geqq 1$; in the case $0<x<1$ we note that $A_1(x, a)=0$ whenever $a\leqq\sqrt{x}$ and that $A_2(x, a)=0$ whenever $(1-a)\leqq\sqrt{x}$, so that

$$F(x) = 1 - \int_{\sqrt{x}}^1 A_1(x, a)da - \int_0^{1-\sqrt{x}} A_2(x, a)da.$$

Substituting for $A_1(x, a)$ and $A_2(x, a)$ from (2) and (3) and evaluating the integrals yields the following expression for $F(x)$ when $x>0$;

$$F(x) = \begin{cases} -2x(\log x + 1) + \frac{1}{3}(1 + 8x^{3/2}) & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geqq 1. \end{cases}$$

As $x\to 0^+$, it is seen that $F(x)\to 1/3$, so that by continuity $F(0)=1/3$. It is interesting to note that this can be obtained also by a combinatorial argument: There are $3!=6$ ways in which the three random variables X_1, X_2, and X_3 can be ordered. Since these random variables are independent and identically distributed, the six orderings are equally likely. But only two of the orderings are favorable to the event $(X_2-X_1)(X_3-X_1)\leqq 0$, namely the orderings $X_2\leqq X_1\leqq X_3$ and $X_3\leqq X_1\leqq X_2$. Therefore $F(0)=2/6=1/3$.

Let us now examine the case $x<0$. Again we consider the unit square in the

X_2-X_3 plane. As in the case of positive x, $F_1(x, a)$ is the total area of the shaded regions in Figure 2; note that the areas of the two regions are equal by symmetry. Also as in the case $x>0$, these regions can disappear depending on the relative magnitudes of x and a. However, in contrast to the case of positive x, by symmetry both regions must be present or both must be absent.

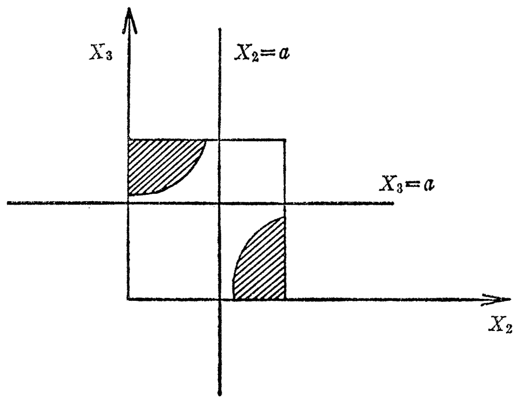

FIG. 2. The shaded region is $\{(X_2, X_3):(X_2-a)(x_3-a)\leqq x\}$ when x is negative.

Consider the upper region. The intersection of the upper branch of the hyperbola with the top of the square is at $X_2=a+x/(1-a)$ and its intersection with the X_3-axis is at $X_3=a-x/a$. This region (hence both regions) will disappear when either $a-x/a>1$ or $a+x/(1-a)<0$. It is not hard to see that these conditions are, in fact, equivalent: the regions will disappear whenever $x<a(a-1)$. Hence $F_1(x, a)=0$ if $x<a(a-1)$; if $x\geqq a(a-1)$, then

$$\begin{aligned} F_1(x, a) &= 2\int_{a-x/a}^{1}\left(a+\frac{x}{X_3-a}\right)dX_3 \\ &= 2(-x\log(-x)+x\log(a(1-a))+a(1-a)+x). \end{aligned} \tag{4}$$

Now a is restricted to the interval $[0, 1]$, and on this interval $a(a-1)$ has the minimum value of $-1/4$; thus if $x<-1/4$, the inequality $x<a(a-1)$ is automatically satisfied so that $F_1(x, a)=0$ and so $F(x)=0$. Assume then that $x\geqq -1/4$. In this case, the inequality $x\geqq a(a-1)$ is equivalent to the following inequality:

$$\tfrac{1}{2}(1-\sqrt{1+4x})\leqq a\leqq \tfrac{1}{2}(1+\sqrt{1+4x}).$$

Therefore if $-1/4\leqq x<0$, we have that

$$F(x)=\int_0^1 F_1(x, a)da=\int_{\frac{1}{2}(1-\sqrt{1+4x})}^{\frac{1}{2}(1+\sqrt{1+4x})} F_1(x, a)da.$$

Substituting in our expression for $F_1(x, a)$ from (4) and performing the integration yields the following expression for $F(x)$ when $-1/4\leqq x<0$:

$$F(x)=2x\log\left(\frac{1+\sqrt{1+4x}}{1-\sqrt{1+4x}}\right)+\frac{1}{3}(1-8x)\sqrt{1+4x}.$$

(Observe that $F(x)\to 1/3$ as $x\to 0^-$.) Putting everything together, we have our final expression for $F(x)$:

$$(5)\quad F(x) = \begin{cases} 0 & \text{if } x \leqq -1/4 \\ 2x\log\left(\dfrac{1+\sqrt{1+4x}}{1-\sqrt{1+4x}}\right) + \dfrac{1}{3}(1-8x)\sqrt{1+4x} & \text{if } -1/4 \leqq x < 0 \\ \dfrac{1}{3} & \text{if } x = 0 \\ -2x(\log x + 1) + \dfrac{1}{3}(1+8x^{3/2}) & \text{if } 0 < x \leqq 1 \\ 1 & \text{if } x \geqq 1. \end{cases}$$

In Figure 3, we have sketched a graph of $F(x)$. Note the vertical tangent at $x=0$.

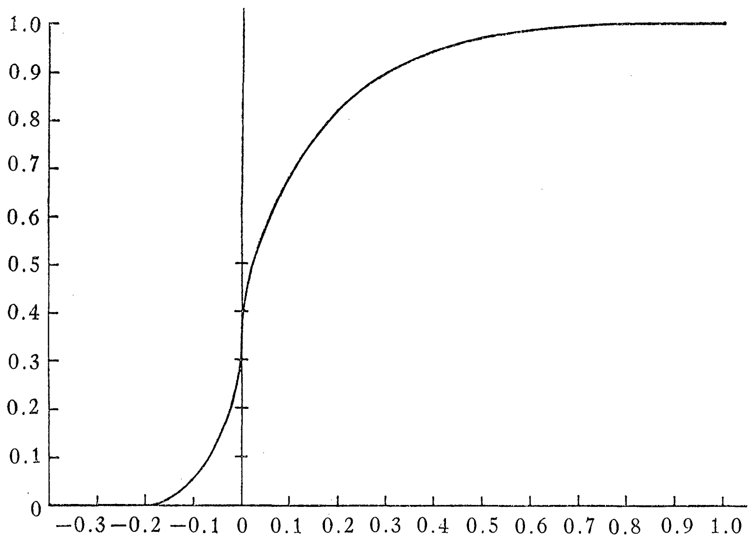

FIG. 3. Graph of $F(x)$, the CDF of $X=(X_2-X_1)(X_3-X_1)$.

Computation of $P(L)$. Obviously $P(L)=P(1/L)$ so that we can assume without loss of generality that $L\geqq 1$. As noted earlier, it is necessary to evaluate the following Riemann-Stieltjes integral:

$$P(L) = 3\int_{-\infty}^{\infty} F(-x/L^2)dF(x).$$

The integrator function $F(x)$ has a vertical tangent at $x=0$, but is otherwise well behaved, i.e., it has a continuous derivative. We can, therefore, write $P(L)$ as an [improper] Riemann integral

$$P(L) = 3\int_{-\infty}^{\infty} F(-x/L^2)F'(x)dx,$$

where the existence of this integral can be verified, even though the integrand is not finite at $x=0$.

The lower limit of integration is always determined by the fact that $F'(x)=0$ if $x<-1/4$. However, the upper limit of integration is determined by the fact that $F'(x)=0$ if $x>1$ in the case $L\geqq 2$, and by the fact that $F(-x/L^2)=0$ if $x>L^2/4$ in the case $1\leqq L\leqq 2$. Thus in general

$$P(L) = 3\int_{-1/4}^{\alpha} F(-x/L^2)F'(x)dx,$$

where $\alpha=\min(1, L^2/4)$.

To evaluate this integral, we shall first split the range of integration into two parts: $-1/4$ to 0 and 0 to α, so that

$$P(L) = 3\int_{-1/4}^{0} F(-x/L^2)F'(x)dx + 3\int_{0}^{\alpha} F(-x/L^2)F'(x)dx. \tag{6}$$

In the first of these integrals, let $y=-x$ and integrate by parts as follows:

$$\int_{-1/4}^{0} F(-x/L^2)F'(x)dx = \int_{0}^{1/4} F(y/L^2)F'(-y)dy$$
$$= \frac{1}{9} + M^2\int_{0}^{1/4} F(-y)F'(M^2y)dy,$$

where we have written $M=1/L$. In the second of the integrals in (6), let $y=x/L^2$ so that

$$\int_{0}^{\alpha} F(-x/L^2)F'(x)dx = L^2\int_{0}^{\beta} F(-y)F'(L^2y)dy,$$

where $\beta=\alpha/L^2=\min(1/L^2, 1/4)$. The problem is thus reduced to the evaluation of the following integral:

$$I(x; A) = \int_{0}^{x} F(-y)F'(A^2y)dy,$$

where $A>0$ and where $x=\min(1/4, 1/A^2)$. In terms of this integral, we see that in general

$$P(L) = \tfrac{1}{3} + (3/L^2)I(1/4; 1/L) + 3L^2I(\beta; L),$$

where $\beta=\min(1/L^2, 1/4)$.

The evaluation of $I(x; A)$ is straightforward, but long and tedious; we omit the details. We note the special case, though, of $x=1/4$:

$$I(1/4; A) = \frac{\pi A}{240} + \frac{47}{900} - \frac{\log A}{15}. \tag{7}$$

In the case $1\leqq L\leqq 2$, we see that $\beta=1/4$, so that $P(L)$ can be written

$$P(L) = \tfrac{1}{3} + (3/L^2)I(1/4; 1/L) + 3L^2I(1/4; L).$$

Substituting in this equation the value of $I(1/4; A)$ from (7), we have the solution for $P(L)$ in the case $1 \leqq L \leqq 2$:

$$\text{(8)} \quad P(L) = \frac{1}{3} + \frac{47}{300}(L^2 + 1/L^2) + \frac{\pi}{80}(L^3 + 1/L^3) - \frac{\log L}{5}(L^2 - 1/L^2).$$

We note two special cases. If $L = 1$, so that the rectangle is a square, we have that

$$P(1) = \frac{97}{150} + \frac{\pi}{40} = 0.72520648 \cdots ;$$

the case $L = 2$ of this equation yields the solution to Hawthorne's problem:

$$P(2) = \frac{1199}{1200} + \frac{13\pi}{128} - \frac{3}{4}\log 2 = 0.79837429 \cdots .$$

In the case $L \geqq 2$, we see that $\beta = 1/L^2$, so that

$$P(L) = \tfrac{1}{3} + (3/L^2)I(1/4; 1/L) + 3L^2 I(1/L^2; L).$$

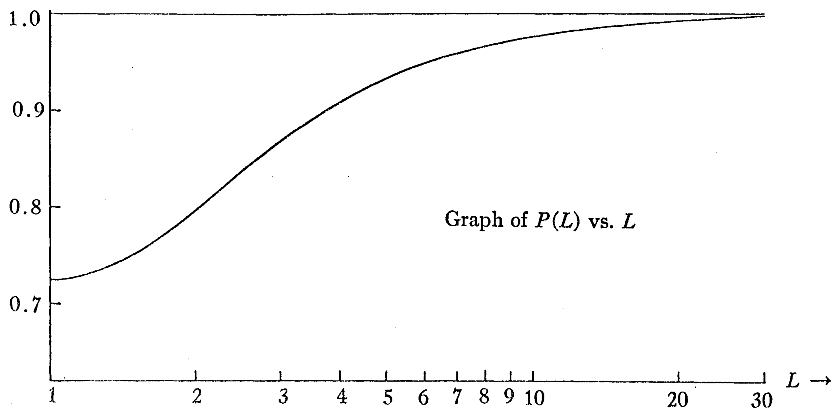

FIG. 4.

Substituting into this the general value of $I(x; A)$ we have our answer for $P(L)$ in the case $L \geqq 2$:

$$P(L) = \frac{1}{3} + \frac{1}{L^2}\left(\frac{\pi}{80L} + \frac{47}{300} + \frac{\log L}{5}\right) + \frac{47L^2}{300} - \frac{L^2 \log L}{5}$$

$$\text{(9)} \qquad + \frac{L^3}{40}\operatorname{Arcsin}\left(\frac{2}{L}\right) + \left(\frac{L^2}{10} - \frac{3}{5L^2}\right)\log\left\{\frac{L + \sqrt{L^2 - 4}}{L - \sqrt{L^2 - 4}}\right\}$$

$$+ \frac{L\sqrt{L^2 - 4}}{150}\left(-31 + \frac{63}{L^2} + \frac{64}{L^4}\right).$$

Note that the last three terms make sense only if $L \geqq 2$; note also that if $L = 2$, then the last two terms vanish, and that $\text{Arcsin}(2/L) = \pi/2$. Thus if we replace L by $\max(L, 2)$ in the last two terms of (9) and in the argument of the Arcsin, then (9) becomes valid even when $1 \leqq L \leqq 2$.

In Figure 4 we have graphed $P(L)$ versus L (on a logarithmic scale).

Acknowledgements. I would like to thank Daniel H. Wagner, Associates for support of this project; I would also like to thank M. L. Golubitsky and W. S. Brainerd for programming assistance. A portion of this work was supported by an ONR Foundation Grant (FY1968).

A greatly abbreviated version of this paper has appeared in Biometrika [3] under the title, *The probability that a random triangle is obtuse.*

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1. Frank Hawthorne, Problem E1150, Amer. Math. Monthly, 62 (1955) 40.
2. C. S. Ogilvy, Tomorrow's Math, Oxford University Press, New York, 1962, p. 114.
3. Eric Langford, The probability that a random triangle is obtuse, Biometrika, 56 (1969) 689–690.

THE EXISTENCE OF FINITE BOLYAI-LOBACHEVSKY PLANES

STEVEN H. HEATH, Southern Utah State College

In the study of finite Projective and finite Euclidean geometries one is led to study more general finite geometries. Of recent interest are systems called finite Bolyai-Lobachevsky geometries, that is, finite geometries in which there is more than one line which is parallel (two lines which do not intersect will be called parallel) to a given line through a point not on that line. L. M. Graves [1] has exhibited a particular example of such a system and has raised the question of the existence of others.

As a basis for our discussion we will exhibit systems which satisfy the following axioms, which are somewhat more restrictive than those of Graves, but which have the desired property of being very similar to the axioms generally used to define finite Projective or finite Euclidean geometries.

Axiom 1. If P and Q are two points there is exactly one line containing P and Q.

Axiom 2. If l is any line there is a point P which does not lie on l.

Axiom 3. There are at least two points on every line.

Axiom 4. There exists at least one line.

Axiom 5. Given a point P not on a line l, there are exactly k lines which are parallel to l and pass through P.

From these postulates we can easily prove the following basic theorem:

THEOREM 1. *If there exists one line which contains exactly n points, then*

a) *Every line contains exactly n points.*

b) *There are exactly $n+k$ lines which pass through each point.*

c) *Space contains exactly $(n+k)(n-1)+1$ points.*

d) *Space contains exactly $[(n+k)(n-1)+1][n+k]/n$ lines.*

then the last two terms vanish, and that $\text{Arcsin}(2/L)=\pi/2$. Thus if we replace L by $\max(L, 2)$ in the last two terms of (9) and in the argument of the Arcsin, then (9) becomes valid even when $1 \leqq L \leqq 2$.

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c) *Space contains exactly $(n+k)(n-1)+1$ points.*

d) *Space contains exactly $[(n+k)(n-1)+1][n+k]/n$ lines.*

A finite BL space satisfying Axioms 1–5 will be denoted by the symbol $BL(k, n)$ where k is the number of parallels (Axiom 5) and n is the number of points on each line. (Note that a finite Projective geometry with n points on each line and a finite Euclidean geometry with n points on each line can be denoted respectively as $BL(0, n)$ and $BL(1, n)$. Note also that Theorem 1 with $k=0$ and 1 gives the fundamental results for finite Projective and Euclidean geometries.)

From Theorem 1 part (d) we can easily derive the following nonexistence result:

THEOREM 2. *If $k(k-1)$ is not divisible by n there is no finite $BL(k, n)$ space corresponding to the given values of k and n.*

We note in passing that if $k=2$ (the Hyperbolic case) then $k(k-1)=2$ and the only integer $n \geqq 2$ (Axiom 3) which divides 2 is 2. Hence the only possible finite BL space with 2 parallels is a $BL(2, 2)$, which can easily be constructed.

If $n=2$ we observe from part (c) of Theorem 1, that the number of points in $BL(k, 2)$ is $k+3$. Consequently since each line consists of just 2 points we see that $BL(k, 2)$ is just the set of combinations of $k+3$ distinct objects taken two at a time, which implies that $BL(k, 2)$ exists for each $k \geqq 2$.

E. H. Moore [2] has pointed out that 3-dimensional finite projective geometry with 3 points on each line leads to a solution of the Kirkman's Fifteen School Girl Problem [3]. Any solution of this problem is a BL (4, 3) in our notation. In fact if we consider m-dimensional finite projective geometry and m-dimensional finite Euclidean geometry we can prove the following existence theorem.

THEOREM 3. *There exist finite $BL(k, n)$ spaces for $k=s^{m-1}+s^{m-2}+ \cdots +s^3 +s^2$ and $n=s+1$ and also for $k=s^{m-1}+s^{m-2}+ \cdots +s^3+s^2+1$ and $n=s$ where s is a power of a prime and m is an integer greater than or equal to 3.*

Proof. If s is a power of a prime both m-dimensional finite projective geometry with $s+1$ points on each line and m-dimensional finite Euclidean geometry with s points on each line exist [4].

Now if we consider a system S in which the lines of $PG(m, s)$ or $EG(m, s)$ are the lines of S and in which the points of $PG(m, s)$ or $EG(m, s)$ are the points of S it is clear that Axioms 1–4 are satisfied. All we need now is that this system S satisfies Axiom 5 in each case.

In $PG(m, s)$ we observe that for each point there are $s^{m-1}+s^{m-2}+ \cdots +s^2 +s+1$ lines which pass through this point [4]. Now if l is any line of $PG(m, s)$ and P is any point not on l then l and P determine a unique plane of $PG(m, s)$ in which there are exactly $s+1$ lines passing through P and intersecting l. This leaves exactly $(s^{m-1}+s^{m-2} \cdots +s^2+s+1)-(s+1)=s^{m-1}+s^{m-2}+ \cdots +s^2$ lines passing through P and not intersecting l. For if any of these $s^{m-1}+s^{m-2} + \cdots +s^2$ lines intersect l then they lie in the plane determined by P and l and consequently must be one of the $s+1$ lines already excluded. Hence we see that $BL(s^{m-1}+s^{m-2}+ \cdots +s^2,s+1)$ exists for s a power of a prime.

Similarly in $EG(m, s)$ for a given point there are exactly $s^{m-1}+s^{m-2}+ \cdots +s^2+s+1$ lines on this point. If l is any line and P is any point not on l, P and

l determine a unique plane π. In π there are exactly $s+1$ lines which pass through P of which s intersect l (1 is parallel) and it is clear that these are the only lines through P which intersect l. Thus there are exactly $(s^{m-1}+s^{m-2}+\cdots s^2+s+1)-s=s^{m-1}+\cdots+s^3+s^2+1$ lines which pass through P and do not intersect l and clearly $BL(s^{m-1}+\cdots+s^3+s^2+1, s)$ exists for s a power of a prime.

Using the above finite $BL(k, n)$ spaces, or any other finite $BL(k, n)$ spaces for that matter, we can generate infinitely many finite $BL(k, n)$ spaces.

Consider a set X of Nn distinct objects with n disjoint subsets $X_1, X_2, \cdots X_n$ each containing N distinct objects. Consider $N(N, n)$ matrices of the following form:

$$M^{(r)} = \| M_1, P_{r2}(M_2), \cdots, P_{rn}(M_n)\|$$

where M_i is a fixed column matrix whose elements are the elements of X_i and where $P_{1i}(M_i)=M_i$ and $P_{ri}(M_i)$ is a permutation of the elements of M_i, $r=1, 2, \cdots, N$.

If the matrix system just described has the added property that for two distinct objects, $a\in X_i$ and $b\in X_j$ there exists a unique matrix $M^{(r)}$ such that a and b are elements of exactly one row of $M^{(r)}$ we shall call the system a rectangular system R.

The first question which arises is, "Do such rectangular systems R exist?" and secondly "What is the relationship of these rectangular systems to the existence of finite $BL(k, n)$ spaces?"

The first question will be answered partially later and the second is explained in the following existence theorem:

THEOREM 4. *If $BL(k, n)$ exists and an (N, n) rectangular system R exists where $N=(n+k)(n-1)+1$ then $BL(N+k, n)$ exists.*

Proof. By hypothesis the (N, n) rectangular system R exists. Hence if we let each row of the N matrices $M^{(r)}$, $r=1, 2, \cdots N$, be a line and let the objects of each of the sets X_i be points, it follows that R describes N^2 lines, nN points and furthermore that there are n points on each line. Now since each of the sets X_i contains N points and since the $BL(k, n)$ space exists and contains N points, therefore by any 1:1 correspondence between X_i and the points of $BL(k, n)$ which exists by hypothesis, we can construct a $BL(k, n)$ from each of the sets X_i. We shall show that if we take an (N, n) rectangular system R with points and lines defined as above, together with the lines of the $n-BL(k, n)$ spaces constructed from the sets X_i, this new system of points and lines is a $BL(N+k, n)$.

First we note that Axioms 2, 3, and 4 are clearly satisfied. To show that Axiom 1 is satisfied we need to consider two cases.

Case 1: $a\in X_i$ and $b\in X_j$, $i\neq j$.

Case 2: $a, b\in X_i$.

Proof (1). Since R is a rectangular system, there exists exactly one line which contains a and b. The only other lines which contain a and perhaps b are the lines of the $BL(k, n)$ space constructed from X_i and since $b\notin X_i$, a and b are not on any of these lines, hence Axiom 1 is satisfied in Case 1.

Proof (2). Since a and $b \in X_i$ this implies a and b are elements of the same column of each $M^{(r)}$, hence no line of R contains a and b. Now X_i determines a $BL(k, n)$ and consequently there is exactly one line in this $BL(k, n)$ which contains a and b and we see that Axiom 1 is also satisfied.

We need only show that the number of parallels in Axiom 5 is $N+k$ for the system described by R and the $n-BL(k, n)$ spaces constructed from the sets X_i. We have three cases to consider here.

Case a: Given a line l of R and a point $P \in X_i$, P not on l.

Case b: Given a line l of the $BL(k, n)$ constructed from X_i and a point $P \in X_i$ and not on l.

Case c: Given a line l of the $BL(k, n)$ constructed from X_i and a point $P \notin X_i$ and not on l.

Proof (a). Since l contains exactly n points of which $n-1$ are not in the set X_i and since P is on exactly one line of each $M^{(r)}$, there are exactly $N-(n-1)$ lines which pass through P and are parallel to l in R. Since $P \in X_i$, there are exactly $n+k$ lines which pass through P in the $BL(k, n)$ constructed from X_i. Moreover since there is exactly one point P_i of the line l in X_i then PP_i is the only line of this $BL(k, n)$ intersecting l, we see that there are $N-(n-1)+(n+k-1)=N+k$ lines passing through P and parallel to l.

Proof (b). Since $P \in X_i$ and $l \subset X_i$ this implies there are exactly k lines passing through P which are parallel to l in the $BL(k, n)$ constructed from X_j. The point P is on N lines of R and furthermore none of these N lines intersects l since $P \in X_i$ and $l \subset X_i$. Hence there are exactly $N+k$ lines which pass through P and are parallel to l.

Proof (c). Since $P \in X_j$, $l \subset X_i$, $i \neq j$, there are exactly $n+k$ lines which pass through P in the $BL(k, n)$ space constructed from X_i and are parallel to l. The point P is on N lines of R but since l contains exactly n points and since $l \subset X_i$ there are exactly $(n+k)+(N-n)=N+k$ lines which pass through P and are parallel to l.

Now since Axioms 1, 2, 3, 4, 5 are satisfied with the number of parallels in Axiom 5 being $N+k$, we see that $BL(N+k, n)$ exists, provided R and $BL(k, n)$ exist.

To illustrate this theorem we will show with three simple examples how it can be used. First, as we shall show later, it is an easy matter to construct rectangular systems R of dimension (7, 3), (9, 3), and (13, 3). Also the finite geometries $PG3=BL(0, 3)$, $N=7$; $E-3=BL(1, 3)$, $N=9$; and $BL(3, 3)$ (Graves example) $N=13$, exist. Therefore if we take a (7, 3) rectangular system and 3 $PG3$'s, we can easily construct a $BL(7, 3)$; if we take a (9, 3) rectangular system and 3 $E-3$'s, we can easily construct a $BL(10, 3)$; and if we take a (13, 3) rectangular system and $3-BL(3, 3)$'s we can construct a $BL(16, 3)$.

We note also that since $N>n$ for all k, n, we really need only look for the existence of (N, n) rectangular systems where $N>n$. Our last result is a partial answer to the question of the existence of rectangular systems R.

THEOREM 5. *If N is a power of a prime then (N, n) rectangular systems R exist for each $n \leqq N$.*

Proof. If N is a power of a prime then there exists a finite field F of order N. Let the marks of F be $1, 2, 3, \cdots, N-1, N$ where N is the zero of F and 1 the unit of F. Let

$$M_i \text{ be the column matrix } \left\| \begin{matrix} \overset{i}{x}_1 \\ \overset{i}{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \overset{i}{x}_N \end{matrix} \right\| \text{ where } i \in F, \text{ and}$$

define

$$M^{(r)} = \left\| M_1, P_{r2}(M_2), \cdots, P_{rn}(M_n) \right\|, \qquad r \in F$$

where

$$P_{ri}(M_i) = P_{ri} \left\| \begin{matrix} \overset{i}{x}_1 \\ \cdot \\ \cdot \\ \overset{i}{x}_N \end{matrix} \right\| = \left\| \begin{matrix} \overset{i}{x}_{1+(r-1)i} \\ \cdot \\ \cdot \\ \overset{i}{x}_{N+(r-1)i} \end{matrix} \right\| .$$

With P_{ri} defined it is now an easy matter to verify that the N–$M^{(r)}$'s form a rectangular system R.

We also have the following test for the existence of other (N, n) rectangular systems R, namely, if $(j-1)(r-s) \not\equiv 0 \bmod N \forall i, j, r, s$ where $i<j$, $1 \leqq i, j, \leqq n$ and $s<r$, $1 \leqq r, s, \leqq N$ then R exists. This condition can be derived by considering the cyclic permutation defined as follows:

$$P_{r+1,i+1}(M_{i+1}) = P_{r+1,i+1} \left\| \begin{matrix} a_{1,i+1} \\ \cdot \\ \cdot \\ \cdot \\ a_{N,i+1} \end{matrix} \right\| = \left\| \begin{matrix} a_{N-(m-1),i+1} \\ \cdot \\ \cdot \\ \cdot \\ a_{N-m,i+1} \end{matrix} \right\|$$

where $m \equiv r \cdot i \bmod N$.

In particular, if $n=3$ the only possible values of $(j-i)$ are 1 and 2. Since $r-s<N$ then $1 \cdot (r-s) \not\equiv 0 \bmod N$, and since $N=(2k+7)$ is odd, $2 \cdot (r-s) \not\equiv 0 \bmod N$. Hence $(n, 3)$ rectangular systems R exist for N odd.

Many of the $BL(k, 3)$ spaces presented here are solutions to the general Kirkman's school girls problem. (Note that $PG3$, $BL(3, 3)$ (Graves example) and $BL(12, 3)$, $BL(60, 3)$ (Theorem 3) exist but for these $N=7$, 13, 31, and 127, respectively, and these systems are not considered Kirkman parades since 7, 13, 31, and 127 are not odd multiples of 3.) Ball [3] lists solutions for N an odd multiple of 3 up to 135. By applying Theorem 4 and using the fact that $(n, 3)$ rectangular systems exist for N odd we find that there are solutions to the general problem for $r \cdot 3^s$, where $s=1, 2, 3, \cdots$ and $r=7, 9, 13, 15, 19, 31, 33, 69, 75, 87, 111, 123$, or 127.

The methods given in this paper exhibit the existence of several infinite classes of $BL(k, n)$ spaces. We note however, that in each case $k \geqq n$ for the spaces that we have found. A natural question is, "Do finite $BL(k, n)$ spaces exist for $n>k$? In particular, does the smallest possible $BL(k, n)$ space $(n>k)$, namely $BL(3, 6)$ exist? Also, is there a $BL(k, 3)$ space for each integer k, not ruled out by Theorem 2?" or, "Are there any values of k and n for which $BL(k, n)$ does not exist excepting for those ruled out by Theorem 2?" There is also the question of the existence of other rectangular systems, hence the possible existence of other $BL(k, n)$ spaces, which needs to be answered more fully.

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ANSWERS

A487. $1010_a = 10_b$ when $a+a^3=b$. For example: when $a=2, b=10$ or $a=3, b=30$. Otherwise, if multiplication dots are placed between the last 0 and the preceding 1 on each side of the equality sign, $0=0$ results.

A488. Otherwise it would be identically zero, for since $f(1)=f(1\cdot 1)=f(1)\cdot f(1)$ we must have $f(1)=1$ or 0. If $f(1)\neq 1$, then it must satisfy $f(1)=0$ and $f(m)=f(1)\cdot f(m)=0$.

A489. Since $\int_a^b f(x)dx=0$, there need not exist a centroid. For the discreet case, just consider a couple (two equal and opposite forces).

A490. It is known that the only odd prime divisors of N^2+1 are of the form $4n+1$. Since 67 is a prime which is not of the form $4n+1$ and $x^4+2x^3+5x^2+4x+5=(x^2+x+2)^2+1$ the conclusion follows.

A491. Let c be the length of the hypotenuse and h be the length of the altitude erected on the hypotenuse. Consider the area of the triangle

$$A = ab/2 = hc/2; \quad \text{thus } h = ab/c.$$

Suppose h is an integer. Since c is prime, either $c \mid a$ or $c \mid b$. This is impossible since $a<c$ and $b<c$ as the length of the hypotenuse always exceeds the length of either leg. The result follows from the contradiction.

(Quickies on page 291.)

GENERALIZATIONS OF THE SYLVESTER PROBLEM

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Let S be a finite set in a linear space. A straight line connecting points of S is said to be an *ordinary line* with respect to S if it passes through exactly two points of S.

THEOREM (Sylvester). *To any finite set S in E^2 which is not collinear there is at least one ordinary line.*

Proof. Consider the collection Λ of all straight lines joining distinct points of S and let P be the set of all positive distances of points of S from members of Λ. S being noncollinear, P is nonempty. Since P is finite there is a smallest element d in it, and corresponding to it a point p and a line l such that the perpendicular distance of p from l is d. We claim l is an ordinary line. Suppose not. Then at least three points $q, r, s \in S$ are on l. Let z be the foot of the perpendicular dropped from p onto l. At least one half line of l determined by z contains two points of S. Suppose r and s lie on such a half line with r either coinciding with z or lying in the open interval (z, s). If l' denotes the line through p and s the distance from r to l' is clearly smaller than d against the choice of l. This shows l to be an ordinary line as asserted.

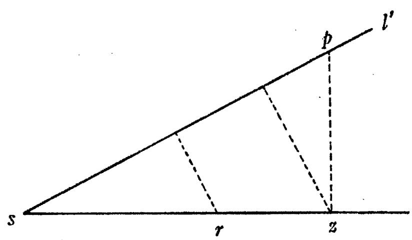

FIG. 1.

The above theorem is a restatement in the affirmative of a question posed in 1893 by J. J. Sylvester [15] which remained unanswered for about forty years. A number of solutions were published in the 'thirties and 'forties. Among the solvers we might mention: Coxeter [3], Dirac [4], Erdös [8], Gallai [9], Kelly and Motzkin (cf. also [9] for solutions by Steinberg and Steenrod). The proof presented here is due to L. M. Kelly. It is rather simple even though it uses a metric argument in a situation involving incidence of points and lines.

To consider some of the possible generalizations let us introduce the notion of an ordinary k-flat and an elementary k-flat.

DEFINITION. *A k-flat (i.e., a translate of a k-dimensional subspace of a linear space) α is said to be an ordinary k-flat of a set S if $S\cap\alpha$ spans α and a point $u\in S$ exists such that $(S\sim\{u\})\cap\alpha$ is contained in a $(k-1)$-flat. It is called an elementary k-flat if no proper subset of $S\cap\alpha$ spans α.*

In E^3, for example, an ordinary plane $\alpha\subset E^3$ contains at least 3 points of the set which are noncollinear and all but one are collinear. An elementary plane is one containing precisely 3 noncollinear points.

In E^2 both an ordinary hyperplane (here a 1-flat) and an elementary hyperplane are an ordinary line.

Motzkin, in his important paper [**14**] conjectured that every finite set spanning E^n has an ordinary hyperplane. Special cases were proved by him, and later by Balomenos, Bonnice, and Silverman [**1**]. Independently, S. Hansen, a Danish high school teacher, succeeded in proving the conjecture in its entirety [**11**].

That elementary hyperplanes may fail to exist is seen from the following simple example. In E^3 let S be a set consisting of six points a_1, a_2, a_3; b_1, b_2, b_3; with the two triples disposed on two skew lines. In any choice of 3 points the plane through them must contain one of the triples so that it cannot be an elementary flat.

It is interesting then to note that Hansen's result implies that if S is a finite set which spans E^{2k} then an elementary k-flat always exists. For $k=1$ this statement is identical with the Sylvester theorem. For $k=2$ we use Hansen's theorem to conclude that a 3-flat α (i.e., a hyperplane) exists which is spanned by $S\cap\alpha$ and a point $u\in S$ in α can be found such that $(S-\{u\})\cap\alpha$ lies in a plane $\beta\subset\alpha$. Now $S\cap\beta$ is not collinear. Hence, by the Sylvester theorem an ordinary line $\lambda\subset\beta$ can be found. Clearly λ together with u span an elementary plane or 2-flat. A similar argument can be used to establish by induction the general case ($k=1, 2, \cdots$).

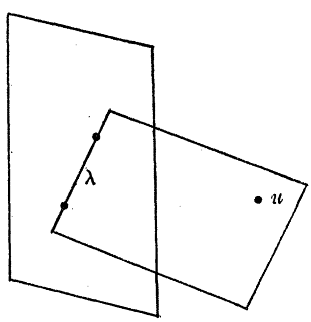

FIG. 2.

A somewhat more general result along with other corollaries to the Hansen's theorem are contained in a joint paper of the writer and W. E. Bonnice [**2**].

Another generalization originated with the idea of Grünbaum of considering finite sets of sets rather than finite sets of points. He proved [**10**] that given a finite collection of two or more connected, compact disjoint sets in E^2 whose union is not contained in a straight line there is always a straight line intersecting exactly two members of the collection.

Connectedness and compactness, both topological properties, give this topic a new flavor, one which seems to belong to the area of linear topological spaces. This direction has been repeatedly undertaken by Herzog, Kelly and myself and far more general results than that of Grünbaum are now available [**5**], [**6**], [**7**], [**12**].

Still within the scope of linear space theory without any topology is the problem, first suggested by Herzog and Kelly, of considering sets of finite sets rather than sets of points. Thus the question is whether if given a finite family of two or more nonempty finite disjoint sets of points in E^n such that the union is not a subset of a straight line, does there exist a straight line which intersects exactly two of them? In E^2 one finds rather easily counterexamples of which we sketch two (see Figures 3 and 4; points marked by integer i belong to set i).

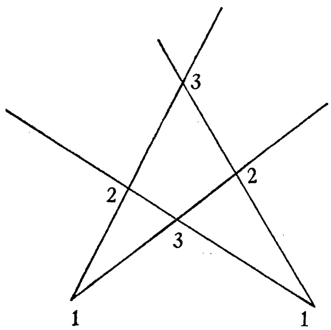

FIG. 3.

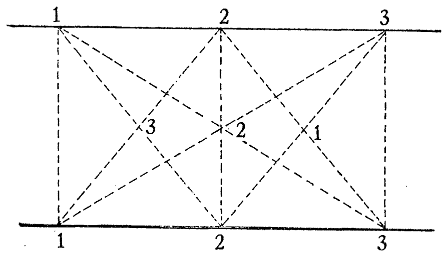

FIG. 4.

In E^3 too, an example of a family as above whose union spans E^3 is known for which a line as described does not exist [7]. Here it is somewhat easier to describe the example in question by embedding E^3 in P^3. No other projectively distinct example of such a family of sets in P^3 is known and it is a nontrivial open question whether there can be such an example. In light of these examples it might be interesting to state and prove the fact that in E^4 (and E^n, $n \geqq 4$) no example of this type is to be found. More specifically (cf. [7]) we have

THEOREM. *Given a finite family* $\{A_1, A_2, \cdots, A_n\}$ *of two or more disjoint nonempty finite sets such that* $\bigcup_1^n A_i$ *spans* E^4 *there is always a straight line which intersects exactly two members of the family.*

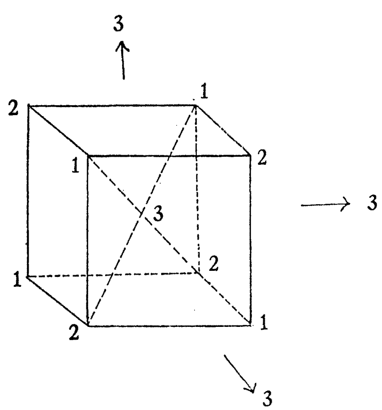

FIG. 5.

The proof of this theorem is based on an observation, made by Motzkin in his paper in the TAMS mentioned before, which we state now.

LEMMA. *Let $S \subset E^3$ be a finite set in E^3 spanning E. Then a plane $\pi \subset E^3$ and two straight lines l_1, l_2 exist such that $l_1 \cup l_2 \subset \pi$, $l_1 \cap l_2$ belongs to S, $l_1 \sim (l_1 \cap l_2)$ and $l_2 \sim (l_1 \cap l_2)$ both contain points of S and $(\pi \sim (l_1 \cup l_2)) \cap S = \varnothing$.*

This lemma follows immediately from the Sylvester theorem. Indeed let $s \in S$ be an arbitrary point of S and consider a plane σ not through s such that no line connecting s to any other point of S is parallel to σ. (Such a plane clearly exists.) Project the set $S \sim \{s\}$ centrally from s onto σ. Clearly the image is finite and not collinear. Hence an ordinary line λ can be found in σ. It is now clear that the plane through λ and s may serve as π (and the two lines through s and $\lambda \cap S$ may serve as l_1 and l_2).

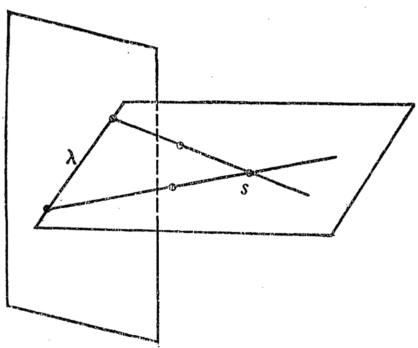

FIG. 6.

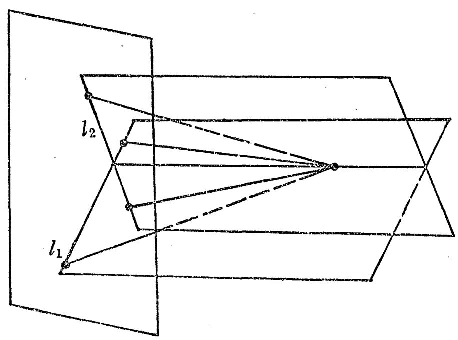

FIG. 7.

Proof of the theorem. Let a be an arbitrary point of A_1 and α_1 a hyperplane through a with the property that no point of $\cup_2^n A_i$ is in α_1. Project $\cup_2^n A_i$ from a into a hyperplane α which is parallel to and distinct from α_1. If the image is contained in a plane $\beta \subset \alpha$ then β and a span together a 3-dimensional flat in E^4 and therefore a point $a' \in A$, must lie outside this flat. This point and any point of $\cup_2^n A_i$ will then determine a straight line as desired, i.e., one which intersects exactly two members of the family. If, on the other hand, this is not so then the image spans α and the preceding lemma is applicable. Let then $\pi \subset \alpha$, l_1, $l_2 \subset \pi$ have the properties stated in that lemma with $S =$ image of $\cup_2^n A_i$. The point a and the plane π span together a 3-dimensional flat and we will exhibit a line as required in this space. Consider the two planes λ_1, λ_2 spanned by l_1 and a, respectively l_2 and a. If a point of A_1 is to be found in the above 3-flat outside the union of the above planes then this point and any point whose image is $l_1 \cap l_2$ will clearly determine a line as required. If not, let $b \in \cup_2^n A_i$ be any point whose image is on $l_1 \sim (l_1 \cap l_2)$. If the line through a and b contains no additional point of that set or only points of one set distinct from A_1 then this line is as desired. Otherwise we have an additional point c on that line belonging to another set. We may now assume that points from two distinct sets of $\{A_2, \cdots, A_n\}$ are also present on the line joining a with a projection on $l_2 \sim (l_1 \cap l_2)$. But then the line through a point on the first line and a point on the second line from two distinct sets of the above family is clearly as required.

In closing we would like to point out one of the many problems which are open in the area. As in the original Sylvester theorem if one proceeded from an ordinary line to questions about ordinary and elementary k-flats one might here formulate questions concerning the existence of flats intersecting a certain number $k=3, 4 \cdots$ of members of the family. Whilst the question for singletons is completely settled the one for sets is open.

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AN APPROACH TO TRIGONOMETRIC INEQUALITIES

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Given triangle $A_1A_2A_3$ with $P \in \text{Int}(A_1A_2A_3)$. Let x_k and p_k represent the distances from P to A_k and to the side opposite A_k respectively. It is known [1] that the following inequalities are equivalent to the Erdös inequality, $x_1+x_2+x_3 \geqq 2(p_1+p_2+p_3)$,

(a) $$x_1p_1 + x_2p_2 + x_3p_3 \geqq 2(p_1p_2 + p_1p_3 + p_2p_3)$$

(b) $$p_1^{-1} + p_2^{-1} + p_3^{-1} \geqq 2(x_1^{-1} + x_2^{-1} + x_3^{-1})$$

(c) $$x_1x_2 + x_1x_3 + x_2x_3 \geqq 2(x_1p_1 + x_2p_2 + x_3p_3)$$

(d) $$x_1x_2 + x_1x_3 + x_2x_3 \geqq 4(p_1p_2 + p_1p_3 + p_2p_3).$$

line through a point on the first line and a point on the second line from two distinct sets of the above family is clearly as required.

In closing we would like to point out one of the many problems which are open in the area. As in the original Sylvester theorem if one proceeded from an ordinary line to questions about ordinary and elementary k-flats one might here formulate questions concerning the existence of flats intersecting a certain number $k=3, 4 \cdots$ of members of the family. Whilst the question for singletons is completely settled the one for sets is open.

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AN APPROACH TO TRIGONOMETRIC INEQUALITIES

HAROLD EHRET, Wisconsin State University, Platteville

Given triangle $A_1A_2A_3$ with $P\in\text{Int}(A_1A_2A_3)$. Let x_k and p_k represent the distances from P to A_k and to the side opposite A_k respectively. It is known [1] that the following inequalities are equivalent to the Erdös inequality, $x_1+x_2+x_3\geqq 2(p_1+p_2+p_3)$,

(a) $$x_1p_1+x_2p_2+x_3p_3\geqq 2(p_1p_2+p_1p_3+p_2p_3)$$

(b) $$p_1^{-1}+p_2^{-1}+p_3^{-1}\geqq 2(x_1^{-1}+x_2^{-1}+x_3^{-1})$$

(c) $$x_1x_2+x_1x_3+x_2x_3\geqq 2(x_1p_1+x_2p_2+x_3p_3)$$

(d) $$x_1x_2+x_1x_3+x_2x_3\geqq 4(p_1p_2+p_1p_3+p_2p_3).$$

The purpose of this paper is to indicate elementary proofs of the results (a)–(d) and to use these results to unify some common trigonometric inequalities.

There is an elementary proof of (a) available [1].

To show (a)⇒(b) construct circles A_1A_2P, A_2A_3P and A_3A_1P and let A_3', A_1' and A_2' be the points diametrically opposite P in these circles. We have then, see [2], the relations

$$\text{(e)} \qquad PA_3' = \frac{x_1x_2}{p_3}, \qquad PA_1' = \frac{x_2x_3}{p_1} \quad \text{and} \quad PA_2' = \frac{x_3x_1}{p_2}.$$

In triangle $A_1'A_2'A_3'$ with interior point P, *by* (a)

$$A_1'P\cdot x_1 + A_2'P\cdot x_2 + A_3'P\cdot x_3 \geqq 2(x_1x_2 + x_1x_3 + x_2x_3).$$

Substituting relations (e) the result is

$$\frac{x_1x_2x_3}{p_1} + \frac{x_1x_2x_3}{p_2} + \frac{x_1x_2x_3}{p_3} \geqq 2(x_1x_2 + x_1x_3 + x_2x_3),$$

or equivalently,

$$p_1^{-1} + p_2^{-1} + p_3^{-1} \geqq 2(x_1^{-1} + x_2^{-1} + x_3^{-1}).$$

To obtain (c) apply (b) to triangle $A_1'A_2'A_3'$, then

$$x_1^{-1} + x_2^{-1} + x_3^{-1} \geqq 2(A_1'P^{-1} + A_2'P^{-1} + A_3'P^{-1})$$
$$= 2\left(\frac{p_1}{x_2x_3} + \frac{p_2}{x_1x_3} + \frac{p_3}{x_1x_2}\right).$$

Multiplying by $x_1x_2x_3$ we obtain result (c). From (a) and (c) we obtain (d) using the transitive property.

If P is the incenter of an arbitrary triangle $A_1A_2A_3$, $p_i=r$ and $x_i=r\csc A_i/2$. By (b):

$$\frac{3}{r} \geqq \frac{2}{r}\left(\sin\frac{A_1}{2} + \sin\frac{A_2}{2} + \sin\frac{A_3}{2}\right)$$

so that

$$\text{(f)} \qquad \sin\frac{A_1}{2} + \sin\frac{A_2}{2} + \sin\frac{A_2}{2} \leqq \frac{3}{2}.$$

From (d):

$$r^2\left(\csc\frac{A_1}{2}\csc\frac{A_2}{2} + \csc\frac{A_1}{2}\csc\frac{A_3}{2} + \csc\frac{A_2}{2}\csc\frac{A_3}{2}\right) \geqq 4(3r^2)$$

or

$$\csc\frac{A_1}{2}\csc\frac{A_2}{2} + \csc\frac{A_1}{2}\csc\frac{A_3}{2} + \csc\frac{A_2}{2}\csc\frac{A_3}{2} \geqq 12$$

and

(g) $$\sin\frac{A_3}{2}+\sin\frac{A_2}{2}+\sin\frac{A_1}{2}\geqq 12\sin\frac{A_1}{2}\sin\frac{A_2}{2}\sin\frac{A_3}{2}\cdot$$

The transitive property applied to (g) and (f) gives us the result:

(h) $$\sin\frac{A_1}{2}\sin\frac{A_2}{2}\sin\frac{A_3}{2}\leqq\frac{1}{8}\cdot$$

Since in a triangle $A_1A_2A_3$

$$\cos A_1+\cos A_2+\cos A_3=1+4\sin\frac{A_1}{2}\sin\frac{A_2}{2}\sin\frac{A_3}{2},$$

we have also

(i) $$\cos A_1+\cos A_2+\cos A_3\leqq\frac{3}{2}\cdot$$

Apply the arithmetic-geometric mean inequality to cos A_1 cos A_2 cos A_3, then:

(j) $$\cos A_1\cdot\cos A_2\cdot\cos A_3\leqq\frac{(\cos A_1+\cos A_2+\cos A_3)^3}{27}\leqq\frac{3/2^3}{27}=\frac{1}{8}\cdot$$

The identity $\cos A=2\cos^2 A/2-1$ applied to (i) gives

(k) $$\cos\frac{A_1}{2}+\cos^2\frac{A_2}{2}+\cos^2\frac{A_3}{2}\leqq\frac{9}{4}\cdot$$

If the root-mean square and A.M.-G.M. inequalities are used with (k) the results are:

$$\frac{\left(\cos\frac{A_1}{2}+\cos\frac{A_2}{2}+\cos\frac{A_3}{2}\right)^2}{9}\leqq\frac{\cos^2\frac{A_1}{2}+\cos^2\frac{A_2}{2}+\cos^2\frac{A_3}{2}}{3}\leqq\frac{9}{12}$$

or

(l) $$\cos\frac{A_1}{2}+\cos\frac{A_2}{2}+\cos\frac{A_3}{2}\leqq\frac{3\sqrt{3}}{2}$$

and

$$\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}\leqq\frac{\left(\cos^2\frac{A_1}{2}+\cos^2\frac{A_2}{2}+\cos^2\frac{A_3}{2}\right)^3}{27}\leqq\frac{9/4^3}{27},$$

that is,

(m) $$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\leqq\frac{3\sqrt{3}}{8}\cdot$$

Finally, since $(\sum_{i=1}^{3} a_i)^2 \geqq 3(a_1a_2+a_1a_3+a_2a_3)$ for all real numbers a_i, we have with the aid of (f)

$$3\left(\sin\frac{A_1}{2}\sin\frac{A_2}{2}+\sin\frac{A_1}{2}\sin\frac{A_3}{2}+\sin\frac{A_2}{2}\sin\frac{A_3}{2}\right.$$
$$\leqq\left(\sin\frac{A_1}{2}+\sin\frac{A_2}{2}+\sin\frac{A_3}{2}\right)^2\leqq\frac{9}{4}\cdot$$

This gives us the inequality

$$\text{(n)}\qquad \sin\frac{A_1}{2}\sin\frac{A_2}{2}+\sin\frac{A_1}{2}\sin\frac{A_3}{2}+\sin\frac{A_2}{2}\sin\frac{A_3}{2}\leqq\frac{3}{4}\cdot$$

This approach requires only a high school mathematics background and includes most of the trigonometric inequalities required in an introduction to geometric inequalities. Other specializations of results (a)–(d) and the Erdös inequality should provide more accessible proofs for other geometric inequalities since they are of a very general nature. For example, if P is the Brocard point of a triangle, we may use the Erdös inequality to show immediately that the Brocard angle w is less than 30°.

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ON THE CONSTRUCTION OF THE REAL NUMBER SYSTEM

W. MORAN, University of Liverpool, and J. S. PYM, University of Sheffield

1. The construction of the real number system begins with one simple system (the rationals) and ends with another. The student who is presented with this construction may therefore enquire what exactly has been done, and why. The answers to these questions may not be as simple as they appear (in particular, we will suggest that the aim is not to prove the existence of the real numbers in any sense) and the arguments which follow are presented in the hope of provoking discussion of the topic.

2. Before we begin, we should indicate what we mean by 'the construction of the real number system'. Briefly, it is that, starting from any system satisfying the axioms of the rationals (i.e., any archimedean linearly ordered field) we can form by the method of Dedekind cuts (or of Cauchy sequences, and so on) a system satisfying the axioms of the reals (i.e., a complete archimedean linearly ordered field). We are discussing any of the methods which might be used in an undergraduate course.

3. The usual situation in mathematics is that the construction of an object

Finally, since $(\sum_{i=1}^{3} a_i)^2 \geqq 3(a_1a_2+a_1a_3+a_2a_3)$ for all real numbers a_i, we have with the aid of (f)

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$$\leqq\left(\sin\frac{A_1}{2}+\sin\frac{A_2}{2}+\sin\frac{A_3}{2}\right)^2\leqq\frac{9}{4}\cdot$$

This gives us the inequality

$$\text{(n)}\qquad \sin\frac{A_1}{2}\sin\frac{A_2}{2}+\sin\frac{A_1}{2}\sin\frac{A_3}{2}+\sin\frac{A_2}{2}\sin\frac{A_3}{2}\leqq\frac{3}{4}\cdot$$

This approach requires only a high school mathematics background and includes most of the trigonometric inequalities required in an introduction to geometric inequalities. Other specializations of results (a)–(d) and the Erdös inequality should provide more accessible proofs for other geometric inequalities since they are of a very general nature. For example, if P is the Brocard point of a triangle, we may use the Erdös inequality to show immediately that the Brocard angle w is less than 30°.

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is taken as a proof of its existence, as for example in the case of the completion of a metric space. But in the case of the reals, there are those who, in spite of the usual construction, wish to qualify the assertion that the whole set of real numbers exists. For example, they might say that certain real numbers (e.g., the rationals and π) exist in a stronger sense than others. On the other hand, the man who wishes to do ordinary differentiation or integration will inevitably demand the existence of the real number system. If all attempts at constructing the reals from the rationals failed, he would say simply that the rationals did not form a sufficient basis for his work, and he would assume the axioms for the reals instead. The construction of the reals is not made in order to prove their existence; this question is already decided before the proof begins.

4. If that is the case, does constructing the reals serve any purpose at all? As working mathematicians, we usually believe that all we do is contained within set theory. If we take this view, we can regard the proof as saying that the real number system can be constructed within (any of the usual formalizations of) set theory. And this is an important result: it states that set theory is adequate to deal with (say) undergraduate analysis.

5. Of course, the proof is not presented to undergraduates in this light. They are told that it is possible to obtain the real numbers from something simpler or more fundamental. But the construction does not use only the simpler system of rationals, it also uses methods of set theory, in particular it considers (say) all Dedekind cuts. Now the assertions that the real numbers exist and that the set of all Dedekind cuts exists are roughly equivalent. Indeed, it is not possible to describe easily all such cuts in terms of rationals (briefly, there is only a countable number of predicates which can be applied to the countable set of rationals, and so only a countable number of reals can be described in this way—e.g., as roots of algebraic equations) but each can be described by giving the real number it defines. To put the same point in an informal way, the existence of Dedekind cuts is similar to the power set axiom, and is a very sophisticated (and even perhaps ambiguous) idea, while the completeness axiom, which is the main stumbling block in the axiom system for the reals, does look obvious when the reals are viewed as infinite decimals. Although the rational number system is simpler than the real, the methods used in the construction are no simpler than the reals themselves.

6. There is a sense in which set theory is more fundamental than the theory of real numbers; the former includes the latter, and the converse is not true. But there is another sense in which the real number system is the more fundamental. No one learns mathematics by starting with axiomatic set theory. He learns first about counting, fractions and decimals, and he comes to believe in real numbers before he has a clear idea of what a set is (and he may never discover what 'set' means to a logician). We might say that the theory of real numbers is 'psychologically prior'. And this is really to put the point of paragraph 3 in an informal way.

7. Set theory is fundamental in the sense that most of what mathematicians wish to say can be expressed in its terms. But it does not follow that set theory is, or ought to be, the one and only foundation for mathematics. It could be that set theory enjoys its present privileged position because of the way the subject has developed; much more effort has been put into solving its problems than into a search for an alternative, though it is possible that category theory will produce a new approach. Of course, any two foundations for mathematics must be in many ways equivalent, for they must carry the same structure, but in a different foundation the concept of set might not be fundamental. We have in mind the situation of projective geometry: it is fruitless to ask which of the synthetic or algebraic approaches is the true one or the more fundamental.

8. One further point should be mentioned. The application of many of the arguments given above is not limited to a discussion of the real number system: is there not something suspicious about this? Let us attempt to apply such an argument to the Hahn-Banach theorem (any other result, say the construction of the Lebesgue measure, would do). If it were discovered that set theory contained an essential flaw, what would happen to this result? We argued that the real number system would remain unharmed, but as the Hahn-Banach theorem depends on so much set theory, we could hardly expect this to happen. Much would depend on the nature of the flaw, but we could imagine a situation in which some mathematicians rejected the theorem, while others argued that its consequences were such that they were still prepared to use it as a reasonable hypothesis. Again, applied mathematicians and physicists would be unmoved by evidence of a crack in the foundations—they have always preferred to be guided by their experience and the usefulness of their results. In other words, acceptance of a result as true is not merely a question of reading a proof (indeed, who would be prepared to give an absolute guarantee that a mistake had not been made) but is also a question of experience.

9. What conclusions can be drawn about teaching the construction of the real number system? Even at present, it is not a question of teaching the student an important method of construction, for Cauchy sequences can be given in a course on metric spaces, and Dedekind cuts in a course on partially ordered sets. Therefore it must be the fact that the reals can be constructed which is regarded as important. In this case, if the above arguments are accepted, it must be made clear that the student is not being shown where real numbers come from, but is being shown that the theory (a vague 'set theory') in which he is doing his mathematics includes the theory of real numbers. It follows that it ought not to be an early part of his course (for when he arrives he has little idea of what a theory is) and should not be part of an elementary analysis course (for this might give the impression that the subject depended on set theory, which we have tried to show is false). Ideally, it should form part of a course in set theory, where its true position as a comment on the theory could be recognized.

We would like to thank our colleagues at the Universities of Sheffield and Liverpool whose disagreement—and occasional agreement—has helped to bring this essay to its present form.

THE TRUTH-VALUE OF $\{\forall, \exists, P(x, y)\}$: A GRAPHICAL APPROACH

ERNEST A. KUEHLS, University of Akron

A picture or a graph in mathematics is worth many words. Suppose that the universal set of discourse is the set of real numbers, Re. The truth-value of a quantified two-variable proposition on Re may be determined by a graphical analysis. A two-variable proposition on Re is a statement containing two variables such that Re is the replacement set for each of the variables. The following list defines the various symbols and terms which will be used in this discussion.

(1) : is translated "represents."
(2) $P(x, y)$: a two-variable proposition on Re.
(3) $\forall x \forall y$ is translated "For all x and y."
(4) $\exists x \exists y$ is translated "There is an x and a y such that."
(5) $\forall x \exists y$ is translated "For each x there is a y such that."
(6) $\exists y \forall x$ is translated "There is a y such that for all x."
(7) A graph will be referred to as a *shading*.

Example 1. Determine the truth-value of each of the following statements if $P(x, y): x^2 > y$.

(a) $\forall x \forall y[P(x, y)]$ (b) $\exists x \exists y[P(x, y)]$ (c) $\forall x \exists y[P(x, y)]$
(d) $\forall y \exists x[P(x, y)]$ (e) $\exists x \forall y[P(x, y)]$ (f) $\exists y \forall x[P(x, y)]$

In order to determine the truth-value of each of the preceding statements via a graphical analysis, we first indicate the shading of $P(x, y)$, i.e., the graph of the truth-set of $P(x, y)$, on an xy-coordinate plane (Figure 1).

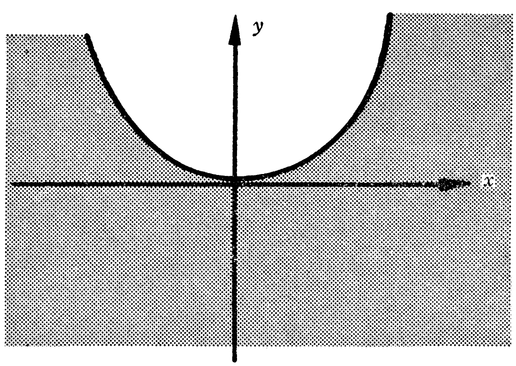

FIG. 1. (The parabola is not included in the shading.)

The truth-value of each of the statements is now determined as follows:

(a*) $\forall x \forall y[P(x, y)]$ is a true statement if and only if the *entire* xy-coordinate plane is shaded. Hence, statement (a) is *false*.

(b*) $\exists x \exists y[P(x, y)]$ is a true statement if and only if there is *some* shading on the xy-coordinate plane. Hence, statement (b) is *true*.

(c*) $\forall x \exists y[P(x, y)]$ is a true statement if and only if *each vertical line* makes contact with the shading. Statement (c) is *true.*

(d*) $\forall y \exists x[P(x, y)]$ is a true statement if and only if *each horizontal line* makes contact with the shading. Statement (d) is *true.*

(e*) $\exists x \forall y[P(x, y)]$ is a true statement if and only if there is *a vertical line* which lies *wholly* in the shading. Hence, statement (e) is *false.*

(f*) $\exists y \forall x[P(x, y)]$ is a true statement if and only if there is *a horizontal line* which lies *wholly* in the shading. Statement (f) is *true.*

Example 2. Determine the truth-value of each of the statements in Example 1 if $P(x, y): y+x=y$.

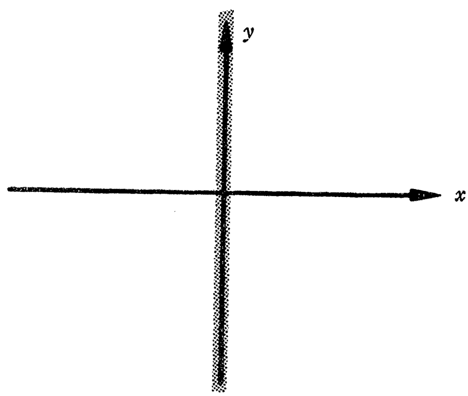

FIG. 2.

The shading of $P(x, y)$ is just the y-axis (Figure 2). The graphical analysis, as outlined in Example 1, clearly indicates that statements (b), (d), and (e) are the only true statements.

The difference between "$\exists x \forall y$" and "$\forall y \exists x$" is lucidly indicated by the graphical approach. "$\exists x \forall y$" demands a *fixed* x whereas "$\forall y \exists x$" does *not* demand a fixed x.

THE GRAM-SCHMIDT PROCESS IS NOT SO BAD!

ANTHONY E. HOFFMAN, State University of New York, College at Geneseo

Three objections to the Gram-Schmidt process were presented in a recent paper [**2**, page 203]. The first was referred to as a personal opinion relative to the elegance of the process and I will not comment on it. The object here is to give a presentation that will remove the objection that the process is hard for the student to remember. In fact, it is hoped that after seeing this presentation even the average student could derive the formulas from basic ideas in a few minutes. By considering the same example as in [**2**] we wish to at least raise the question of whether the process presented there is arithmetically less cumbersome than the one described here.

(c*) $\forall x \exists y [P(x, y)]$ is a true statement if and only if *each vertical line* makes contact with the shading. Statement (c) is *true.*

(d*) $\forall y \exists x [P(x, y)]$ is a true statement if and only if *each horizontal line* makes contact with the shading. Statement (d) is *true.*

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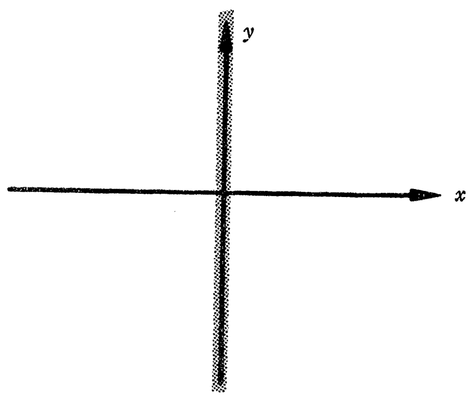

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Our presentation is based on the elementary properties of the inner product encountered in the standard calculus course. See, for example, [1, page 514]. The student knows that $(\boldsymbol{\alpha}, \boldsymbol{\beta})=\alpha\beta\cos\theta$ where θ is the angle between the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and as usual, α denotes the length of $\boldsymbol{\alpha}$. The student can note that $\alpha\cos\theta$ is the projection of $\boldsymbol{\alpha}$ on $\boldsymbol{\beta}$. If we desire to write $\boldsymbol{\alpha}=\boldsymbol{\mu}+\boldsymbol{\delta}$ where $\boldsymbol{\mu}$ is parallel to $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ is orthogonal to $\boldsymbol{\beta}$ we can note that the length of $\boldsymbol{\mu}$ should be $\alpha\cos\theta$. Since $\boldsymbol{\beta}/\beta$ is a unit vector parallel to $\boldsymbol{\beta}$, we can write:

$$\boldsymbol{\mu} = (\alpha\cos\theta)(\boldsymbol{\beta}/\beta) = \frac{\alpha\beta\cos\theta}{\beta\beta}\,\boldsymbol{\beta} = \frac{(\boldsymbol{\alpha}, \boldsymbol{\beta})}{(\boldsymbol{\beta}, \boldsymbol{\beta})}\,\boldsymbol{\beta}. \tag{1}$$

Then we have $\boldsymbol{\delta}=\boldsymbol{\alpha}-((\boldsymbol{\alpha}, \boldsymbol{\beta})/(\boldsymbol{\beta}, \boldsymbol{\beta}))\boldsymbol{\beta}$.

To develop our procedure for applying the Gram-Schmidt process we will make repeated use of (1). The general problem is to obtain an orthogonal basis $\{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots, \boldsymbol{\varepsilon}_n\}$ for a subspace S when we are given a basis $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ for S.

With the observation that the abstract inner product is a generalization of the scalar product of calculus we proceed as follows: Choose $\boldsymbol{\varepsilon}_1=\boldsymbol{\alpha}_1$. To obtain $\boldsymbol{\varepsilon}_2$ we subtract from $\boldsymbol{\alpha}_2$ the component of $\boldsymbol{\alpha}_2$ which is parallel to $\boldsymbol{\varepsilon}_1$. Using (1) we obtain

$$\boldsymbol{\varepsilon}_2 = \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha}_2, \boldsymbol{\varepsilon}_1)}{(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1)}\,\boldsymbol{\varepsilon}_1.$$

To obtain $\boldsymbol{\varepsilon}_3$ we subtract from $\boldsymbol{\alpha}_3$ the component of $\boldsymbol{\alpha}_3$ parallel to $\boldsymbol{\varepsilon}_1$ and the component of $\boldsymbol{\alpha}_3$ parallel to $\boldsymbol{\varepsilon}_2$. Using (1) again we obtain

$$\boldsymbol{\varepsilon}_3 = \boldsymbol{\alpha}_3 - \frac{(\boldsymbol{\alpha}_3, \boldsymbol{\varepsilon}_1)}{(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1)}\,\boldsymbol{\varepsilon}_1 - \frac{(\boldsymbol{\alpha}_3, \boldsymbol{\varepsilon}_2)}{(\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_2)}\,\boldsymbol{\varepsilon}_2.$$

In general,

$$\boldsymbol{\varepsilon}_t = \boldsymbol{\alpha}_t - \frac{(\boldsymbol{\alpha}_t, \boldsymbol{\varepsilon}_1)}{(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1)}\,\boldsymbol{\varepsilon}_1 - \frac{(\boldsymbol{\alpha}_t, \boldsymbol{\varepsilon}_2)}{(\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_2)}\,\boldsymbol{\varepsilon}_2 - \cdots - \frac{(\boldsymbol{\alpha}_t, \boldsymbol{\varepsilon}_{t-1})}{(\boldsymbol{\varepsilon}_{t-1}, \boldsymbol{\varepsilon}_{t-1})}\,\boldsymbol{\varepsilon}_{t-1}.$$

By routine induction one can verify that $\{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots, \boldsymbol{\varepsilon}_n\}$ is an orthogonal basis for S.

All that is required of a student to be able to derive these "working" equations for the Gram-Schmidt process is an understanding of (1) and of what he is trying to do.

The example of [2] is considered here to allow comparison of the two procedures. Recall that E is the space of polynomials of degree three or less where the inner product is defined by

$$(p, q) = \int_{-1}^{1} p(t)q(t)dt.$$

The choice of basis was $\{1, t, t^2, t^3\}$. Then:

$$\boldsymbol{\varepsilon}_1 = 1$$

$$\varepsilon_2 = t - \left(\int_{-1}^{1} t dt \Big/ \int_{-1}^{1} dt\right) 1 = t$$

$$\varepsilon_3 = t^2 - \left(\int_{-1}^{1} t^2 dt \Big/ \int_{-1}^{1} dt\right) 1 - \left(\int_{-1}^{1} t^3 dt \Big/ \int_{-1}^{1} t^2 dt\right) t$$

$$= t^2 - 1/3$$

$$\varepsilon_4 = t^3 - \left(\int_{-1}^{1} t^3 dt \Big/ \int_{-1}^{1} dt\right) 1 - \left(\int_{-1}^{1} t^4 dt \Big/ \int_{-1}^{1} t^2 dt\right) t$$

$$- \left(\int_{-1}^{1} \{t^5 - t^3/3\} dt \Big/ \int_{-1}^{1} \{t^4 - 2t^2/3 + 1/9\} dt\right) (t^2 - 1/3).$$

$$= t^3 - (0)1 - ((2/5)/(2/3))t - (0)(t^2 - 1/3)$$

$$= t^3 - (3/5)t.$$

It should be noted that if we are given any generating set $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ for a nonzero subspace S, then the nonzero vectors in the set $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k\}$ constructed above constitute an orthogonal basis for S.

References

1. R. E. Johnson and F. L. Kiokemeister, Calculus with Analytic Geometry, Allyn and Bacon, Boston, 1969.

2. J. H. Staib, An alternative to the Gram-Schmidt process, this MAGAZINE, 42 (1969) 203–205.

A DIFFERENTIAL-FUNCTIONAL EQUATION; THE COMPLEX CASE

DANIEL P. GIESY, University of Southern California

0. Introduction. In another paper [1] we consider the differential functional equation

$$D_z f(g(z)) = g(z), \tag{1}$$

where z is a real number and D_z is ordinary real differentiation obtaining all solutions on connected domains of real numbers. In this paper we shall consider the same equation where now z is a complex variable and D_z is ordinary complex differentiation. We give necessary and sufficient conditions that a given f or g participate in a solution and how to calculate one given the other. The first part of the paper handles (1) locally, the latter part, globally.

In addition to whatever intrinsic value the solution of (1) has, this paper serves two purposes; first, it illustrates the application of a large part of the content of a first graduate course in complex analysis, and second, in conjunction with [1], it provides one more illustration of the striking difference between real and complex differentiation.

There are two striking similarities between the real and complex case: the

$$\varepsilon_2 = t - \left(\int_{-1}^{1} t\,dt \Big/ \int_{-1}^{1} dt\right) 1 = t$$

$$\varepsilon_3 = t^2 - \left(\int_{-1}^{1} t^2 dt \Big/ \int_{-1}^{1} dt\right) 1 - \left(\int_{-1}^{1} t^3 dt \Big/ \int_{-1}^{1} t^2 dt\right) t$$

$$= t^2 - 1/3$$

$$\varepsilon_4 = t^3 - \left(\int_{-1}^{1} t^3 dt \Big/ \int_{-1}^{1} dt\right) 1 - \left(\int_{-1}^{1} t^4 dt \Big/ \int_{-1}^{1} t^2 dt\right) t$$

$$- \left(\int_{-1}^{1} \{t^5 - t^3/3\} dt \Big/ \int_{-1}^{1} \{t^4 - 2t^2/3 + 1/9\} dt\right) (t^2 - 1/3).$$

$$= t^3 - (0)1 - ((2/5)/(2/3))t - (0)(t^2 - 1/3)$$

$$= t^3 - (3/5)t.$$

It should be noted that if we are given any generating set $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ for a nonzero subspace S, then the nonzero vectors in the set $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k\}$ constructed above constitute an orthogonal basis for S.

References

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In addition to whatever intrinsic value the solution of (1) has, this paper serves two purposes; first, it illustrates the application of a large part of the content of a first graduate course in complex analysis, and second, in conjunction with [1], it provides one more illustration of the striking difference between real and complex differentiation.

There are two striking similarities between the real and complex case: the point 0 in the domain of f plays an exceptional role and the inverse function of g plays a key role in the solution.

1. Necessary conditions for solution.

1.1. THEOREM. *Suppose $D_z f(g(z)) = g(z)$ for all $z \in D$ a domain. Then,*

(a) *$g \equiv 0$ on D; or*

(b) (1) *g is regular in D,*

(2) *f is regular in $g(D)$,*

(3) *$g'(z) \neq 0$ for each $z \in D$,*

(4) *$f'(w) \neq 0$ for each $w \in g(D) \setminus \{0\}$, and*

(5) *if $0 \in g(D)$, then $f'(0) = 0$ and $f''(0) \neq 0$.*

Proof. Let $h(z) = f(g(z))$. If g is constant, then h is constant in D, so $g = h' \equiv 0$ on D. Thus (a) holds.

Now assume g is nonconstant in D. By hypothesis, h is regular in D and $g = h'$, so g is regular in D, which establishes (1), and since $g' \not\equiv 0$, the zeros of g' are isolated. For each $z_0 \in D$ with $g'(z_0) \neq 0$, there exists a function G regular at $g(z_0)$ such that $g(G(w)) = w$ in some neighborhood of $w_0 = g(z_0)$. Then, near w_0, $f(w) = f(g(G(w))) = h(G(w))$, so f is regular at w_0. Hence (3)$\Rightarrow$(2). We now establish (3).

Suppose $g'(z_0) = 0$ and let $w_0 = g(z_0)$. Since z_0 is an isolated zero of g' and g is open at z_0, the previous paragraph shows that f is regular in a deleted neighborhood of w_0. Since $f(g(\cdot))$ is differentiable at z_0, it is continuous there, and again since g is open at z_0, f is continuous at w_0 and hence by the theorem on removable singularities, f is regular at w_0. Therefore,

$$f'(g(z))g'(z) = g(z)$$

in a neighborhood of z_0. Since $g'(z_0) = 0$, $g(z_0) = 0$ and $f'(g(\cdot)) = g/g'$ has a simple zero at z_0. But g has a zero of order at least 2 at z_0, so $f'(g(\cdot))$ does also, a contradiction. This establishes (3).

By (2), $f'(g(z))g'(z) = g(z)$ for all $z \in D$. If $g(z) \neq 0$, then $f'(g(z)) \neq 0$ establishing (4). If $g(z) = 0$, since by (3) $g'(z) \neq 0$, then $f'(g(z)) = 0$. Differentiating again, we get

$$f''(g(z))[g'(z)]^2 + f'(g(z))g''(z) = g'(z),$$

so since $f'(g(z)) = 0$ and $g'(z) \neq 0$, we must have $f''(g(z)) \neq 0$. This establishes (5).

The following theorem establishes a connection between the f and g of a solution to (1) which will be useful in establishing uniqueness of solutions.

1.2. THEOREM. *Suppose $D_z f(g(z)) = g(z)$ in a neighborhood of z_0 and $g \not\equiv 0$ there. Then g is invertible in a neighborhood of z_0 with inverse function G defined in a neighborhood of $w_0 = g(z_0)$, and $f'(w) = wG'(w)$ in a neighborhood of w_0.*

Proof. By (3) of Theorem 1.1, $g'(z_0) \neq 0$ so the inverse function G exists. By (1) and (2) of Theorem 1.1, $f'(g(z))g'(z) = g(z)$, so in a neighborhood of w_0,

$$f'(w) = f'(g(G(w))$$

$$= g(G(w))/g'(G(w))$$
$$= wG'(w),$$

since $1/g'(G(w)) = G'(w)$.

2. Existence of local solutions. In this section we show that the necessary conditions for solution of (1) given in Theorem 1.1 are in fact sufficient for existence of local solutions and we give the degree of uniqueness of solutions.

2.1. THEOREM. *Let g be regular at z_0 with $g'(z_0) \neq 0$. Then there exists a function f such that $D_z f(g(z)) = g(z)$ in a neighborhood of z_0. f is unique up to an additive constant, and all functions of the form $f(\cdot)+c$ are solutions, with this g, of (1) in a neighborhood of z_0.*

Proof. Let $w_0 = g(z_0)$ and let G be the inverse of g regular at w_0, and let h be an antiderivative of g in a neighborhood of z_0. Let $f(w) = h(G(w))$. Then $f(g(z)) = h(z)$, so $D_z f(g(z)) = g(z)$. Since by Theorem 1.2, $f'(w) = wG(w)$, f' is uniquely determined by g, so f is unique up to an additive constant. It is immediate that if (f, g) is a solution to (1), then so is $(f(\cdot)+c, g)$ for any complex constant c.

2.2. THEOREM. *Let f be regular at w_0. If $w_0 \neq 0$, suppose $f'(w_0) \neq 0$ while if $w_0 = 0$, suppose $f'(w_0) = 0$ and $f''(w_0) \neq 0$. Then for each z_0 there exists a unique g regular at z_0 such that $g(z_0) = w_0$ and $D_z f(g(z)) = g(z)$ in a neighborhood of z_0. If g_1 is the unique function for z_1, $g_1(z) = g(z - z_1 + z_0)$ (g and g_1 differ by a translation by $z_1 - z_0$).*

Proof. Pick z_0. Let $H(w) = f'(w)/w$. The hypotheses insure that H is regular at w_0 and $H(w_0) \neq 0$. Let G be the function determined by the conditions $G(w_0) = z_0$ and $G'(w) = H(w)$ in a neighborhood of w_0. Then G has an inverse g regular at z_0 with $g(z_0) = w_0$. Then in a neighborhood of z_0,

$$\begin{aligned} D_z f(g(z)) &= f'(g(z))g'(z) \\ &= g(z)G'(g(z))g'(z) \\ &= g(z), \end{aligned}$$

since $f'(w) = wG'(w)$ and $G'(g(z)) = 1/g'(z)$

By Theorem 1.2, G' is determined by f', hence G and therefore g is uniquely determined by the choice of z_0.

It is an elementary calculation to show that if (f, g) satisfies (1) in a neighborhood of z_0 with $g(z_0) = w_0$, then $(f, g(\cdot - z_1 + z_0))$ satisfies (1) in a neighborhood of z_1 with $g(z_1 - z_1 + z_0) = w_0$, so by the uniqueness result above, $g_1(z) = g(z - z_1 + z_0)$.

3. Global solutions. The following question naturally arises: If D is a domain and g is regular on D and for each $z \in D$, $g'(z) \neq 0$, does there exist a function f defined on $g(D)$ such that (f, g) satisfy (1) on D? One can sometimes find such f (or g in the converse problem; see 3.4 and 3.5) but not in all cases. For example, if $g(z) = z^2$ and D is the plane without the origin, then one local solution to (1) is $f(w) = w^{3/2}/3$. But this does not have a single-valued branch in $g(D)$. In fact, using a function like $g(z)=\sin z$ so $g(0)=g(\pi)$ but $g(z)\not\equiv g(z+\pi)$ in a neighborhood of 0, even if D is a simply connected domain containing 0 and π and avoiding the zeros of g' there is no function f defined on $g(D)$ so that (f, g) solves (1) on D.

However, there is a natural situation in which there are always global solutions to (1) using a given f or g, namely, if we consider analytic functions. Here, we use "analytic function" to mean an equivalence class of power series where two power series are equivalent if there is a finite chain of power series starting with the one and ending with the other and such that each element of the chain (except the first) is a direct rearrangement of the previous one. (See Hille [2] for details concerning analytic functions and analytic continuation.) By "element of an analytic function" we mean ambiguously either one of the power series in the analytic function or the function defined by that power series in its region of convergence. Since a function which is regular at a point is given by its Taylor's series in a neighborhood of that point, and if a function is regular in a domain, all of its Taylor's series centered in that domain are in the same analytic function, and an analytic function is determined by any one of its elements, we will identify a function regular in a domain with the analytic function which contains the Taylor's series of that function centered in the given domain.

3.1 THEOREM. *If (f, g) solves (1) in a neighborhood of z_0, and if g^* is another element of g in a neighborhood of z^*, then there exists an element f^* of f defined in a neighborhood of $w^*=g^*(z^*)$ such that (f^*, g^*) solves (1) in a neighborhood of z^*.*

Proof. If g is constant, hence by Theorem 1.1, $g\equiv 0$, then $g^*\equiv 0$, so take $f^*=f$. Otherwise, continue g analytically to g^* along a path $C=\{z(t): 0\leqq t\leqq 1\}$ which avoids the isolated zeros of g'. Continue f along $g(C)$. The set of t for which the continuation of f to $g(z(t))$ exists and gives a local solution to (1) with this g is either $[0, 1]$ in which case we are done or $[0, a)$ for some $a>0$; since the original solution is in a neighborhood of z_0 which means for f in a neighborhood of $w_0=g(z_0)$ since g is open at z_0. But if the latter case holds, we can use Theorem 2.1 to find a solution f_a for (1) at $z(a)$ using this g, and we adjust the ambiguous constant so $f_a(g(z(b)))=f(g(z(b)))$ for some $b<a$. Therefore by the uniqueness of the solution at $z(b)$, $f_a\equiv f$ on a neighborhood of $g(z(b))$ so f_a continues f analytically to $z(a)$ as a solution to (1) with this g. This is a contradiction showing that this case cannot occur.

3.2. THEOREM. *If (f, g) solves (1) in a neighborhood of z_0 and if f^* is another element of f in a neighborhood of w^*, then there exists an element g^* of g defined at some point z^* such that $g^*(z^*)=w^*$ and (f^*, g^*) solves (1) in a neighborhood of z^*.*

Proof. If g is constant, hence by Theorem 1.1, $g\equiv 0$, pick $g^*\equiv 0$. Otherwise g and hence f are nonconstant. Let $w_0=g(z_0)$ and let G be the inverse of g defined in a neighborhood of w_0. Continue f from w_0 to w^* along a path C avoiding 0 and the zeros of f'. Hence, by standard techniques, $G'(w)=f'(w)/w$ can be continued analytically along C and is never 0 on C, so G can be continued analytically along C and is never 0 on C, so G and g can be continued analytically along

C and $G(C)$ respectively, so that corresponding elements are inverse functions. Hence if G^* is the continuation of G to w^* along C, $z^* = G^*(w^*)$, and if g^* is the continuation of g to z^* along $G(C)$, g^* and G^* are inverses, $f^{*\prime}(w) = wG^{*\prime}(w)$, and so (f^*, g^*) solves (1) in a neighborhood of z^*.

3.3. REMARKS. Solutions to (1) can be classified as trivial ($g \equiv 0$, f defined at 0, otherwise arbitrary) and nontrivial (f and g both nonconstant analytic functions). Call nonconstant analytic functions f and f_1 F-equivalent if some element of f and some element of f_1 differ by a constant (so each element of f differs from a corresponding element of f_1 by this constant) and call nonconstant analytic functions g and g_1 G-equivalent if some element of g differs from some element of g_1 by a translation (so each element of g differs from a corresponding element of g_1 by the same translation). If (f, g) is a pair of nonconstant analytic functions, say (f, g) solves (1) if some element of f and some element of g form a solution of (1) in some domain (so there is a correspondence between all elements of f and all elements of g (not necessarily one-to-one in either direction since different elements of g can be translates of one another and different elements of f can differ by constants) such that corresponding elements form local solutions of (1)). Then if (f, g) solves (1), f and f_1 are F-equivalent, and g and g_1 are G-equivalent, then (f_1, g_1) solves (1). So, finally, the relation "(f, g) solves (1)" induces a one-to-one correspondence between the set of F-equivalence classes and the set of G-equivalence classes.

For example, if $g(z) = e^z$, $G(w) = \log w$, $wG'(w) = 1$ so $f'(w) = 1$ and $f(w) = w$. The F-equivalence class of f is $\{w \to w + c\colon \text{all } c\}$ and the corresponding G-equivalence class is $\{z \to e^{z+a}\colon \text{all } a\}$.

If $f(w) = w^3$, $f'(w)/w = 3w = G'(w)$, so $G(w) = \frac{3}{2}w^2$ and $g(z) = (\frac{2}{3}z)^{1/2}$. The F-equivalence class and G-equivalence class are $\{w \to w^3 + c\}$ and $\{z \to (\frac{2}{3}(z+a))^{1/2}\}$.

3.4. THEOREM. *If D is a domain, g is regular in D, g' has no zeros in D, and $g(D)$ is simply connected, then there is a function f regular on $g(D)$ such that (f, g) solves* (1) *on D.*

Proof. This is an immediate consequence of the monodromy theorem and proof of Theorem 3.1 since by that proof elements of f solving (1) with this g at points in D can be continued to one another by paths entirely in $g(D)$.

3.5. THEOREM. *Let f be defined in a domain R such that at each point of R, f satisfies the restrictions of Theorem 2.2 and such that the image D of R under the map G where $G'(w) = f'(w)/w$ is simply connected. Then there exists a function g regular on D such that (f, g) satisfies* (1) *on D.*

Proof. Use the monodromy theorem and the proof of Theorem 3.2 as in the previous proof.

References

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A SHORT PROOF OF THE URYSOHN METRIZATION THEOREM

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In this note the Urysohn metrization theorem is proved directly by using the lattice of topologies rather than by embedding in a cube. Such a proof is useful in a first topology course in which one wants to give some idea of the extent to which topological spaces are more general than pseudometric spaces but does not have time for other applications of the Tychonov cube.

LEMMA. *The least upper bound (in the lattice of all topologies on X) of a countable family of pseudometric topologies on X is a pseudometric topology.*

Proof. For each n in N (N denotes the set of all natural numbers) let d_n be a pseudometric on X. We may assume without loss of generality that the d_n-diameter of X is $\leqq 1$ for each n. Define d by $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$. Clearly d is a pseudometric for X.

Let t_n denote the topology associated with d_n and t the topology associated with d. Since $d \geqq 2^{-n} d_n$, it follows that each d_n-sphere of radius ϵ contains the d-sphere of radius $2^{-n}\epsilon$ with the same center, and thus $t_n \subset t$.

We complete the proof by showing that $\bigcup \{t_n : n \in N\}$ is a subbasis for t. Let S be a d-sphere with radius ϵ and center x. We construct a finite number of subbasic sets whose intersection contains x and is contained in S. Choose k in N so that $\sum_{n=k+1}^{\infty} 2^{-k} < \epsilon/2$, and for $i = 1, 2, \cdots, k$ let S_i be the d_i-sphere with center x and radius $\epsilon/2$. We show that $S_1 \cap S_2 \cap \cdots \cap S_k$ has the desired properties. Clearly it contains x; to show that it is contained in S, let y be an element of it. Then

$$\begin{aligned} d(x, y) &= \sum_{n=1}^{k} 2^{-n} d_n(x, y) + \sum_{n=k+1}^{\infty} 2^{-n} d_n(x, y) \\ &< \sum_{n=1}^{k} 2^{-n}\epsilon/2 + \sum_{n=k+1}^{\infty} 2^{-n} < \epsilon/2 + \epsilon/2. \end{aligned}$$

This completes the proof.

URYSOHN METRIZATION THEOREM. *A regular, second countable, T_1-space is metrizable.*

Proof. Let $(x, \mathfrak{I})$ be a topological space satisfying the hypotheses. We actually construct the desired metric. As in the usual proof (e.g., see [1], page 125) there is a countable set $\{f_n : n \in N\}$ of continuous real-valued functions on X which distinguishes points and closed sets in X. Each f_n induces a pseudometric d_n on S with d_n-diameter $\leqq 1$, viz., $d_n(x, y) = \min\{|f_n(x) - f_n(y)|, 1\}$.

Let d, t_n, t be defined as in the proof of the lemma. Then d is a pseudometric. Since $\{f_n\}$ separates points, d is a metric. To show that the associated metric topology t coincides with $\mathfrak{I}$, it suffices to show that $t_n \subset \mathfrak{I} \subset t$ for each n (because $t = \text{lub } t_n$ by the lemma). The first inclusion follows because f_n is $\mathfrak{I}$-continuous and the second because $\{f_n\}$ distinguishes points and closed sets

Remark. This method of proof can also be used in the more general Nagata-Smirnov metrization theorem.

Reference

1. J. L. Kelley, General Topology, Van Nostrand, New York. 1960.

COMPLEX NUMBER ALGEBRA AS A SIMPLE CASE OF HEAVISIDE OPERATIONAL CALCULUS

D. H. MOORE, The University of Wisconsin—Green Bay

In [1] the author offered the following postulational basis for an algebraic structure called a *heaviside operational calculus*. Let G be an additive Abelian group of operands. Let E be a set of endomorphisms on G. If a nonnull element, U, of G has the property that every element of G is the map of U under some endomorphism in E, we call U a *unit element* of G relative to E. Suppose G and E satisfy the following postulates:

1. The endomorphisms in E are permutable.
2. G contains at least one *unit element*, U, relative to E.
3. Any endomorphism in E that sends a unit element U into a nonnull operand sends every nonnull operand in G into a nonnull operand.

It follows, as shown in [1], that E is an integral domain under the usual definitions of equality, sums, and products for endomorphisms. For some choices of G and E, E will also be a field. G and E (or G and a class of operators associated with E) determine a *heaviside operational calculus*.

For example, let G be the class of entering, sectionally continuous functions of t. The unit step function, $\{u(t)\}$, will serve as a unit operand for postulate 2. Given a member, f, of G we find the following endomorphism on G that sends u into f:

$$\{g(t)\} \quad \text{goes into} \quad \left\{\int_{-\infty}^{\infty} g(t-\tau)df(\tau)\right\}.$$

This resulting class, E, of endomorphisms is in 1—1 correspondence with G, satisfies the three postulates, and so is an integral domain. It is not a field. The resulting calculus is a restricted heaviside calculus for functions of a continuous variable, expressible in terms of the integral operator Q.

Now let G be any two-dimensional vector space over the real numbers. G is then also an additive Abelian group. Let e_1 and e_2 form a basis for G. We will use e_1 as a *unit operand* for postulate 2. Let e be any member of G. We wish to find an endomorphism on G that sends e_1 into e. As a guide in finding it, we think of the familiar vector space, V_2, of elementary geometry and physics, based on $\bar{\imath}$ and $\bar{\jmath}$, whose members are represented by arrows in an xy plane. Let $\bar{v}$ be any member of V_2. The vector obtained by rotating $\bar{v}$ 90° counterclockwise will be called the *complement* of $\bar{v}$ and denoted by $\bar{v}^*$. If $\bar{v}=x\bar{\imath}+y\bar{\jmath}$, then $\bar{v}^*=-y\bar{\imath}+x\bar{\jmath}$.

Remark. This method of proof can also be used in the more general Nagata-Smirnov metrization theorem.

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3. Any endomorphism in E that sends a unit element U into a nonnull operand sends every nonnull operand in G into a nonnull operand.

It follows, as shown in [1], that E is an integral domain under the usual definitions of equality, sums, and products for endomorphisms. For some choices of G and E, E will also be a field. G and E (or G and a class of operators associated with E) determine a *heaviside operational calculus*.

For example, let G be the class of entering, sectionally continuous functions of t. The unit step function, $\{u(t)\}$, will serve as a unit operand for postulate 2. Given a member, f, of G we find the following endomorphism on G that sends u into f:

$$\{g(t)\} \quad \text{goes into} \quad \left\{\int_{-\infty}^{\infty} g(t-\tau)df(\tau)\right\}.$$

This resulting class, E, of endomorphisms is in 1—1 correspondence with G, satisfies the three postulates, and so is an integral domain. It is not a field. The resulting calculus is a restricted heaviside calculus for functions of a continuous variable, expressible in terms of the integral operator Q.

Now let G be any two-dimensional vector space over the real numbers. G is then also an additive Abelian group. Let e_1 and e_2 form a basis for G. We will use e_1 as a *unit operand* for postulate 2. Let e be any member of G. We wish to find an endomorphism on G that sends e_1 into e. As a guide in finding it, we think of the familiar vector space, V_2, of elementary geometry and physics, based on $\bar{\imath}$ and $\bar{\jmath}$, whose members are represented by arrows in an xy plane. Let $\bar{v}$ be any member of V_2. The vector obtained by rotating $\bar{v}$ 90° counterclockwise will be called the *complement* of $\bar{v}$ and denoted by $\bar{v}^*$. If $\bar{v}=x\bar{\imath}+y\bar{\jmath}$, then $\bar{v}^*=-y\bar{\imath}+x\bar{\jmath}$.

We have:

$$\bar{v}^{**} = -\bar{v}$$
$$(\bar{u}+\bar{v})^* = \bar{u}^* + \bar{v}^*.$$

The use of the notion of *complement* of $\bar{v}$ will enable us to escape the need for the word "rotation" in defining endomorphisms. "Rotation" is fitting enough in V_2 but is unnatural in an arbitrary 2-dimensional vector space. Thus, to convert $\bar{\imath}$ into $x\bar{\imath}+y\bar{\jmath}$, without using "rotation", we say "multiply $\bar{\imath}$ by x, multiply $\bar{\imath}^*$ by y, and add."

With this motivation we return to the above arbitrary two-dimensional vector space, G, based on e_1 and e_2. As a member of a basis, e_1 is nonnull. Given $g=ce_1+de_2$, we define g^*, the complement of g, by: $g^*=-de_1+ce_2$. Then

$$e^{**} = -e, \qquad (e+f)^* = e^* + f^*, \qquad e_1^* = e_2, \qquad e_2^* = -e_1.$$

Also, if g is nonnull, then g and g^* are linearly independent. Given $e=ae_1+be_2$, we are now ready to prescribe an endomorphism on G that, in particular, sends e_1 into e. Let g be an arbitrary member of G.
Consider the mapping:

$$g \quad \text{goes into} \quad ag + bg^*.$$

This is a mapping of G into itself that sends e_1 into e. To show that this mapping preserves sums and so is an endomorphism, we have:

$$g+h \quad \text{goes into } a(g+h) + b(g+h)^*$$
$$= \text{map of sum} = (ag+bg^*) + (ah+bh^*) = \text{sum of maps}.$$

We thus have a class, E, of endomorphisms on G, that satisfies postulate 2 by construction. It is routine to check that these endomorphisms are permutable. As shown in [1], the map of e_1, alone determines an entire mapping, and we may speak of *the* mapping in E which sends e_1 into a given operand e. Note that ce_1+de_2 is null if and only if $c^2+d^2=0$. To check postulate 3, suppose $a^2+b^2\neq 0$ and $c^2+d^2\neq 0$. Then the endomorphism that sends e_1 into the nonnull operand $ae_1+be_1^*$ will send the nonnull operand, ce_1+de_2, into

$$a(ce_1+de_2) + b(ce_1+de_2)^* = (ac-bd)e_1 + (ad+bc)e_2.$$

But

$$(ac-bd)^2 + (ad+bc)^2 = (a^2+b^2)(c^2+d^2) \neq 0.$$

Thus all three postulates hold and so the class, E, of endomorphisms is an integral domain. But E in this case is also a field; every nonzero endomorphism has an inverse; the endomorphism that sends e_1 into

$$\frac{a}{a^2+b^2}e_1 - \frac{b}{a^2+b^2}e_2,$$

where $a^2+b^2\neq 0$, is the inverse of the endomorphism that sends e_1 into ae_1+be_2.

To show this we apply these endomorphisms in succession to an arbitrary operand, g:

$$g \to ag + bg^* \to \frac{a}{a^2+b^2}(ag+bg^*) - \frac{b}{a^2+b^2}(ag+bg^*)^*$$

$$= \frac{a^2g + abg^* - abg^* + b^2g}{a^2+b^2} = g.$$

Next we associate operators with these endomorphisms. An operator is an information storage device that stores, encoded, information about what operand is the map of each operand. The notation for an operator is placed in front of the notation for an operand, in multiplicative form; this provides a useful notation for the map of the operand under the mapping with which the operator is associated. In the present case, the endomorphism that sends e_1 into ae_1+be_2 is completely determined by the ordered pair (a, b), and we may use this ordered pair as the associated operator. Then the operator-operand product "$(a, b)g$" denotes the map of g under the endomorphism that sends e_1 into ae_1+be^*, and we have:

$$(a, b)g = ag + bg^*.$$

These ordered pair operators we call *complex numbers* and we define equality, sums, and products for them to reflect equality, sums, and products for the associated endomorphisms.

I. *Equality*:

$$(a, b) = (c, d) \text{ iff}$$

any $g \in G: (a, b)g = (c, d)g$ (by definition)

$$: ag + bg^* = cg + dg^*$$

$$: (a-c)g + (b-d)g^* = 0.$$

Then $(a, b) = (c, d)$ iff $a = c$ and $b = d$, since g and g^* are linearly independent.

II. *Sums*:

$$[(a, b) + (c, d)]g = (a, b)g + (c, d)g \text{ (by definition)}$$

$$= ag + bg^* + cg + dg^*$$

$$= (a+c)g + (b+d)g^* = (a+c, b+d)g.$$

Then $(a, b) + (c, d) = (a+c, b+d)$, from 1.

III. *Products*:

$$[(a, b)\cdot(c, d)]g = (a, b)[(c, d)g] \text{ (by definition)}$$

$$= (a, b)[cg + dg^*]$$

$$= a(cg + dg^*) + b(cg + dg^*)^*$$

$$= (ac - bd)g + (ad + bc)g^*$$

$$= (ac - bd, ad + bc)g.$$

Then $(a, b)\cdot(c, d) = (ac - bd, ad + bc)$, from 1.

Since $(a, 0)g = ag$, we may identify the complex number $(a, 0)$ with the real number a. To obtain the traditional notation, let $(0, 1) = i$. Then

$$i^2 = (-1, 0) = -1, \qquad (a, b) = (a, 0)(1, 0) + (b, 0)(0, 1) = a + bi.$$

Thus complex number algebra follows from the same postulates as heaviside operational calculus, where the operands in the former case are vectors in a two dimensional vector space, and the operands in the latter case are entering, sectionally continuous functions.

Reference

1. D. H. Moore, Algebraic basis of heaviside operational calculus, J. Franklin Inst., 286 (1968) 158–161.

ON SOME SOLUBLE *N*th ORDER DIFFERENTIAL EQUATIONS

MURRAY S. KLAMKIN, Ford Motor Company, Dearborn, Michigan

This note was suggested by problem 3227 [*Amer. Math. Monthly*, Dec., 1927] due to H. Reddick which was to solve the differential equation

$$[D^5 - D - 4/x]y = 0.$$

The published solution was by F. Miller who first shows there are solutions of the form $y = x^n e^{ax}$ where $n = 1$, $a = \pm 1$, $\pm i$. Then by a somewhat lengthy reduction process, he obtains the fifth linearly independent solution in the form

$$xe^{-x}\int\left[e^{2x}\int\left\{e^{-x}\cos x\int(\sec^2 x\int x^{-5}\cos x dx)\, dx\right\}dx\right]dx.$$

In the same issue of the MONTHLY, Reddick in problem proposal 3299 asks for the solution of the following generalization of the above equation:

$$[D^{n+1} - D - n/x]y = 0.$$

A solution is given by R. H. Sciobereti [Feb., 1929]. Although his solution is obtained straightforwardly by contour integration it is also rather lengthy. Additionally, one of the linear independent solutions is not given explicitly but as the definite integral

$$\int_0^\infty e^{-xt}t^{n-1}(t^n + 1)^{-2}dt \quad (x > 0,\ n \text{ odd}).$$

A much simpler solution can be obtained by just letting $y = vx$ which reduces the equation to

$$[xD + n + 1][D^n - 1]v = 0.$$

Whence,

$$[D^n - 1]v = \frac{k}{x^{n+1}} \qquad (k - \text{arbitrary constant}).$$

Since $(a, 0)g = ag$, we may identify the complex number $(a, 0)$ with the real number a. To obtain the traditional notation, let $(0, 1) = i$. Then

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Whence,

$$[D^n - 1]v = \frac{k}{x^{n+1}} \qquad (k - \text{arbitrary constant}).$$

The latter equation can be solved in terms of single integrals by using the operational formula

$$\frac{1}{D^n - 1} \equiv \frac{1}{n} \sum_{r=0}^{n-1} \frac{\omega^r}{D - \omega^r}$$

where ω is a primitive root of $z^n = 1$. Thus,

$$y = \frac{kx}{n} \sum_{r=0}^{n-1} \omega^r e^{x\omega^r} \int \frac{e^{-x\omega^r} dx}{x^{n+1}} .$$

By pairing the conjugate roots of unity, y can be expressed in real form. For example, in problem 3227, we have

$$[D^4 - 1]y/x = \frac{k}{x^5}$$

or

$$y = x\left\{\frac{1}{D^2 - 1} - \frac{1}{D^2 + 1}\right\}\frac{k}{2x^5},$$

giving

$$\frac{y}{x} = \frac{k}{4}\left\{e^x \int x^{-5}e^{-x}dx - e^{-x}\int x^{-5}e^x dx\right\}$$
$$+ \frac{k}{2}\left\{\cos x \int x^{-5} \sin x dx - \sin x \int x^{-5} \cos x dx\right\} .$$

A further extension is given by differential equations of the form

$$[D^{n+2} - D^2 - 2x^{-1}nD - x^{-2}n(n - 1)]y = 0.$$

On letting $y = vx^2$, we obtain

$$[xD + n + 1][xD + n + 2][D^n - 1]v = 0.$$

Then

$$[D^n - 1]v = \frac{A}{x^{n+1}} + \frac{B}{x^{n+2}}$$

and we proceed as before.

All the previous extensions are special cases of the equation

(1) $$[D^{n+r} - (D^r + A_1x^{-1}D^{r-1} + A_2x^{-2}D^{r-2} + \cdots + A_rx^{-r})]y = 0$$

where the constants A_i are determined from the identity

(2) $$\begin{aligned}[D^{n+r} - (D^r + A_1x^{-1}D^{r-1} + A_2x^{-2}D^{r-2} + \cdots + A_rx^{-r})]x^r v \\ \equiv [xD + n + 1][xD + n + 2] \cdots [xD + n + r][D^n - 1]v.\end{aligned}$$

To determine the A_i's, we first expand out

$$[xD + n + 1][xD + n + 2] \cdots [xD + n + r] \equiv x^r D^r + B_1 x^{r-1} D^{r-1} + B_2 x^{r-2} D^{r-2} + \cdots + B_r.$$

Letting $x = e^z$, we get

$$[D + n + 1][D + n + 2] \cdots [D + n + r] \equiv B_r + B_{r-1} D + B_{r-2} D(D-1) + \cdots + D(D-1) \cdots (D - r + 1).$$

By letting D successively take on the values 0, 1, 2, $\cdots r-1$, we find that

$$B_j = j! \binom{r}{j} \binom{n+r}{j}.$$

Now by expanding out the operators on both sides of (2), we get

$$\begin{aligned} & x^r D^{n+r} + 1! \binom{r}{1}\binom{n+r}{1} x^{r-1} D^{n+r-1} + \cdots + r! \binom{r}{r}\binom{n+r}{r} D^n \\ & - \left[x^r D^r + 1! \binom{r}{1}\binom{r}{1} x^{r-1} D^{r-1} + \cdots + r! \binom{r}{r}\binom{r}{r} \right] \\ & - A_1 \left[x^{r-1} D^{r-1} + 1! \binom{r}{1}\binom{r-1}{1} x^{r-2} D^{r-2} + \cdots + (r-1)! \binom{r}{r-1}\binom{r-1}{r-1} \right] \\ & - A_2 \left[x^{r-2} D^{r-2} + 1! \binom{r}{1}\binom{r-2}{1} x^{r-3} D^{r-3} + \cdots + (r-2)! \binom{r}{r-2}\binom{r-2}{r-2} \right] \\ & \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - A_r \\ & \equiv x^r D^{n+r} + B_1 x^{r-1} D^{r+n-1} + \cdots + B_r D^n \\ & \quad - [x^r D^r + B_1 x^{r-1} D^{r-1} + \cdots + B_r]. \end{aligned}$$

Equating coefficients of D^j, we obtain the following triangular set of equations

$$B_j = \sum_{i=0}^{j} (j-i)! \binom{r}{j-i} \binom{r-i}{j-i} A_i. \tag{3}$$

By ordinary induction, it is easy to guess that

$$A_i = i! \binom{n}{i} \binom{r}{i}.$$

To verify this, we merely substitute back in (3), giving

$$B_j = j! \binom{r}{j} \binom{n+r}{j} = \sum_{i=0}^{j} i!(j-i)! \binom{n}{i}\binom{r}{i}\binom{r}{j-i}\binom{r-i}{j-i}.$$

On simplifying, the latter reduces to the well-known identity

$$\binom{n+r}{j} = \sum_{i=0}^{j}\binom{n}{i}\binom{r}{j-1}$$

It now follows that the solution of (1) is given by the solution of

$$[D^n - 1]yx^{-r} = \sum_{i=1}^{r}\frac{k_i}{x^{n+i}}$$

and is solved as before.

SOLUTIONS OF $A^k+B^k=C^k$ IN NONSINGULAR INTEGRAL MATRICES

P. M. GIBSON, University of Alabama in Huntsville

In [1] Bolker considers solutions of the matrix equation $A^k+B^k=C^k$ in nonsingular n-square integral matrices. He remarks that he suspects that no solutions exist when $k=3$ and $n=2$. In this note we show that solutions do exist in this case. More generally solutions exist if k is any positive odd integer and either $n\geqq k$ or n is even.

A matrix A is *involutory* if A^2 is the identity matrix. We denote the direct sum of the matrices A, B by $A \oplus B$.

LEMMA. *Let R be a ring with unity, n a positive even integer, and M the ring of all n-square matrices over R. There exist involutory matrices A, B, C, in M such that $A+B=C$.*

Proof. The matrices

$$A_0=\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix},\qquad B_0=\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix},\qquad C_0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

are involutory with $A_0+B_0=C_0$. If $n=2m$ let A, B, C be equal to the direct sum of m copies of A_0, B_0, C_0, respectively. Then A, B, C are involutory with $A+B=C$.

THEOREM. *Let k, n be positive integers, where k is odd. If $n\geqq k$ or n is even then there exist nonsingular n-square integral matrices A, B, C such that $A^k+B^k=C^k$.*

Proof. First, assume that n is even. By the lemma there exist involutory integral matrices A, B, C such that $A+B=C$. Since these matrices are involutory and k is odd, $A^k+B^k=C^k$. Now assume that n is odd, $n=k+m\geqq k$. Bolker [1] shows that there exist k-square nonsingular integral matrices A_1, B_1, C_1 such that $A_1^k+B_1^k=C_1^k$. If $n>k$ then the first part of the proof shows that there exist m-square nonsingular integral matrices A_2, B_2, C_2 such that $A_2^k+B_2^k=C_2^k$. In this case let

$$A = A_1 \oplus A_2,\qquad B = B_1 \oplus B_2,\qquad C = C_1 \oplus C_2.$$

$$\binom{n+r}{j} = \sum_{i=0}^{j} \binom{n}{i}\binom{r}{j-1}$$

It now follows that the solution of (1) is given by the solution of

$$[D^n - 1]yx^{-r} = \sum_{i=1}^{r} \frac{k_i}{x^{n+i}}$$

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are involutory with $A_0+B_0=C_0$. If $n=2m$ let A, B, C be equal to the direct sum of m copies of A_0, B_0, C_0, respectively. Then A, B, C are involutory with $A+B=C$.

THEOREM. *Let k, n be positive integers, where k is odd. If $n \geqq k$ or n is even then there exist nonsingular n-square integral matrices A, B, C such that $A^k+B^k=C^k$.*

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$$A = A_1 \oplus A_2, \qquad B = B_1 \oplus B_2, \qquad C = C_1 \oplus C_2.$$

If $n=k$ let $A=A_1$, $B=B_1$, $C=C_1$. In both cases A, B, C are nonsingular n-square integral matrices such that $A^k+B^k=C^k$.

Reference

1. E. D. Bolker, Solutions of $A^k+B^k=C^k$ in $n\times n$ integral matrices, Amer. Math. Monthly, 75 (1968) 759–760.

ON DIVISOR FUNCTIONS

R. G. BUSCHMAN, University of Wyoming

The following argument does not seem to be readily available in the textbooks. Suppose that we take N boxes and a sufficient number of counters. On the first pass over the boxes we dribble one counter into each box; on the second pass, one counter into each second box, etc. On the kth pass we dribble one counter into each kth box using $[N/k]$ counters on this pass and we continue until the process terminates. Since a counter thus goes into box m on the kth pass if and only if k divides m, the number of counters in box m is $\tau(m)$, the number of divisors of m. The total number of counters used can be counted either by boxes or by passes and we obtain the identity

$$\sum_{m=1}^{N} \tau(m) = \sum_{k=1}^{N} [N/k].$$

Similar types of arguments can be used to obtain identities for some other number-theoretic functions. As an example, if we dribble k counters into each kth box on the kth pass, we have an identity involving $\sigma(m)$, the sum of the divisors of m,

$$\sum_{m=1}^{N} \sigma(m) = \sum_{k=1}^{N} k[N/k].$$

Application of "summation by parts" will convert this identity to the more familiar form

$$\sum_{m=1}^{N} \sigma(m) = \sum_{k=1}^{N} (1/2)[N/k]([N/k]+1).$$

Further, if we dribble one counter into each p_kth box (p_k being the kth prime), on the kth pass we obtain the identity

$$\sum_{m=1}^{N} \omega(m) = \sum_{p\leq N} [N/p]$$

in which $\omega(m)$ denotes the number of distinct prime divisors of m.

If $n=k$ let $A=A_1$, $B=B_1$, $C=C_1$. In both cases A, B, C are nonsingular n-square integral matrices such that $A^k+B^k=C^k$.

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BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

C *A Second Course in Calculus, 2nd ed.* By Serge Lang. Addison-Wesley, Reading, Mass., 1968. 305 pp. $8.50.

The title of the book describes the contents rather well; it could not be called a book on advanced calculus because of the severe limitations on what is included. There are 306 pages of text, numbered from 317 to 622 as a continuation of the First Course in Calculus by the same author. The chapter numbers, however, are not continued from the first book.

Chapters 1–7 constitute a block of material, beginning with a chapter on vectors in n-space, followed by differentiation of vectors, partial derivatives, the chain rule, the gradient, potential functions, line integrals, higher derivatives and Taylor's formula, and problems in maxima and minima. Chapters 8–12 offer the basic material in linear algebra needed in the final chapters 13–17. These last five chapters are concerned with the Jacobian matrix, multiple integrals, Green's Theorem, Fourier series, and normed vector spaces. (The chapter on normed vector spaces appeared to this reviewer as an anomaly, more appropriately a first chapter of a more advanced book.) Many of the chapters are on the short side, offering just a bare-bones introduction to a subject.

For example, the work on vectors is light on the geometry of the subject, with virtually nothing on the scalar triple product and its application to volumes and to coplanar lines and vectors. The divergence and curl of a vector field are mentioned only in exercises, not in the text itself. Green's theorem is included, but not the divergence theorem or Stokes' theorem. Rotations in 3-space are not treated. The length of a curve and the area of a surface are given scant attention. These limitations are cited not as criticisms but as simple facts that a potential user should keep in mind. They are not criticisms because the author makes no claim to matching one of the 700 or 800 page treatises. But this reviewer found that he wanted to offer his class additional material in dittoed form to augment the text.

It was also necessary to give the class additional problems, not for every section in the book but only in some instances. Generally speaking, the book has substantial problem sets, but not uniformly. The last section in Chapter 5, on the question of the dependence of a line integral on the path of integration, is followed by one problem, and that is just an extension of the theory to take care of the case "when h is negative" in the proof of the central theorem. A similar observation can be made about the balance of problems between applications of the theory and extensions of the theory, with the latter given almost exclusive attention in a few cases.

The book was well received by the students. They learned quickly that the answers in the back of the book are not completely reliable, so that the question

"Is the answer given to problem so-and-so correct?" was often asked by a student. For a second edition the number of typographical errors seemed rather large, including a rather novel one: the title of Chapter 5 is a little different in the Table of Contents than in the text itself.

The book is contemporary in idiom and in spirit. The proofs, many of which are novel and clever, are arranged so that most of the arguments are incisive and brief. The vector approach is central, so that separate arguments for 2-space and 3-space are avoided except where concepts are tied to a specific dimension, such as the cross-product of two vectors. The notation is mostly chosen to facilitate the entire line of argument. However, the use of symbols like A and $(a_1, a_2, \cdots, a_n)$ to denote either a point or a vector was not a happy choice; this notational arrangement caused the students trouble when confronted with a vector from a point A to a point B.

In summary, the book was found to be quite acceptable for classroom use. There are many excellent features and no serious flaws.

IVAN NIVEN, University of Oregon

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College
ASSOCIATE EDITOR, MURRAY S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be legible and submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before May 15, 1971.

PROPOSALS

775. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

$$\text{Prove} \int_0^1 \sqrt[q]{1 - x^p}\, dx = \int_0^1 \sqrt[p]{1 - x^q}\, dx, \quad \text{where } p, q > 0.$$

776.* *Proposed by Robert Siemann, University of Wisconsin, Waukesha.*

Given an ellipse as shown in the diagram. $\overleftrightarrow{AQ}$ is a tangent to it at one of its vertices A. Let P be a point on the ellipse such that $\overset{\frown}{AP} = \overleftrightarrow{AQ}$ corresponding to a

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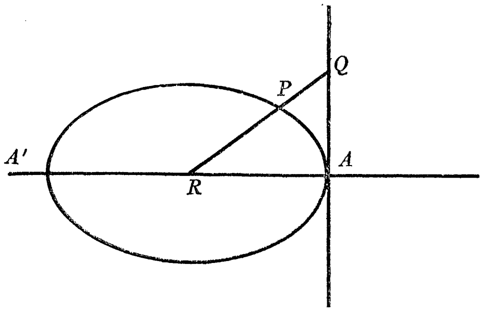


777. *Proposed by Alexandru Lupas, Institutul de Calcul, Cluj, Rumania.*

Let $P_5(A_1, A_2 \cdots A_5)$ be a convex pentagon and let $Q_5(B_1, B_2 \cdots B_5)$ be a convex pentagon which is determined by the line segments A_iA_{i+2}, $i=1, 2, \cdots, 5$ (e.g., $B_1 = A_1A_3 \cap A_5A_2$ and $B_5 = A_4A_1 \cap A_5A_2$). Let Ω be a point which is not in the exterior of Q_5. We denote by r_i the distances of Ω from the sides of P_5, d_i are the distances of Ω from the sides of Q_5 and $R_i = \overleftrightarrow{\Omega A_i}$, $i=1, 2, \cdots, 5$. Prove that

$$3\sum_{i=1}^{5} R_i > 2\sum_{i=1}^{5} r_i + 4\sum_{i=1}^{5} d_i.$$

778. *Proposed by Erwin Just, Bronx Community College.*

Let G be a finite group with subsets $T_1, T_2, \cdots, T_k$ chosen so that $|T_i| = 2^i$. Prove that the smallest order of G for which it is possible that

$$G \neq \prod_{i=1}^{k} T_i \quad \text{is } 3\cdot 2^{k-1}.$$

779. *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers. Show that $\prod_{j-1}^{n} |\sin^j\alpha_j \cos\alpha_j| \leq 1/(n+1)^{(n+1)/2}$, and if $0 \leq \alpha_j \leq \pi/2$ the equality sign holds if and only if

$$\alpha_j = \cos^{-1} 1/\sqrt{j-1}, \quad j = 1, 2, \cdots, n.$$

780. *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let there be given a plane bounded closed convex set with interior points and with boundary of length p. If $p < \pi(2+\sqrt{3})/3$, then one can rotate and translate this set in the plane so that in one position at least it will contain no lattice points.

781. *Proposed by Claude Raifaizen, Bayside, New York.*

If a, b, c, and n are integers and furthermore if $a^n + b^n = c^n$ with either a or b prime or the power of a prime and n an even positive integer, then n does not exceed two.

SOLUTIONS

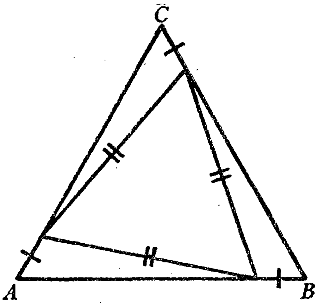
Late Solutions

Andrew N. Aheart, West Virginia State College: **749**; *A. J. Patsche, Rock Island, Illinois:* **750**; *C. F. Pinzka, University of Cincinnati:* **753**.

Nested Equilateral Triangles

754.* [March, 1970] *Proposed by NSF Class at University of California at Berkeley.*

Show that the triangle ABC is equilateral.



I. *Solution by Leon Bankoff, Los Angeles, California.*

Let D, E, F denote the vertices of the inscribed equilateral triangle lying on BC, AC, AB. The proof will make use of the following theorem:

In a triangle ABC, if sides a and b remain constant while the angle between them increases, the side c opposite the variable angle increases. At the same time, if $b<a$, the angle between c and b decreases.

Without loss of generality, assume $C \geqq B \geqq A$. Then

$$C \geqq B \geqq A \Leftrightarrow AB \geqq AC \geqq BC \Leftrightarrow AF \geqq CE \geqq BD \Leftrightarrow A \leqq C \leqq B.$$

It follows that $C=B$, with the consequence that $AB=AC$ and $AF=CE$. Thus triangles AFE and CED are congruent and $C=A$. Hence triangle ABC is equilateral.

This proof is valid only if $CD \leqq ED$, as suggested in the diagram. In the case where $CD \geqq ED$, assume $AF \geqq CE \geqq BD$. Then $\angle AEF \geqq \angle CDE \geqq \angle BFD$ and it follows that $\angle CED \leqq \angle BDF \leqq \angle AFE$ (since angles FED, EDF and DFE are equal). But the latter inequality implies $CE \geqq BD \geqq AF$, establishing an order of magnitude contrary to that of the initial assumption. Hence only the equality signs hold, and $CE=BD=AF$, with the result that triangle ABC is equilateral.

II. *Solution by S. O. Schachter, Temple University.*

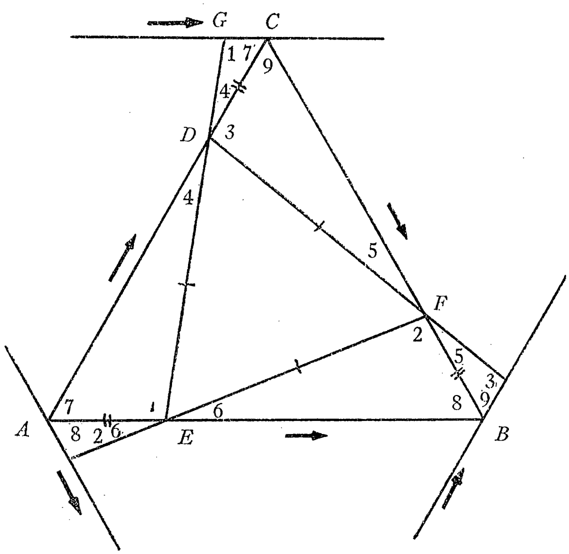
Given: $DF=FE=ED$, $DC=FB=EA$.
Prove: $\triangle ABC$ is equilateral.

Proof: Construct $\overline{BI} \| \overline{AC}$, $\overline{CG} \| \overline{BA}$, $\overline{AH} \| \overline{CB}$. Extend $\overline{ED}$ to $\overline{GC}$, $\overline{FE}$ to $\overline{HA}$,

$\overline{DF}$ to $\overline{IB}$ $\triangle AHE \sim \triangle BFE \sim \triangle AED$, $\triangle BIF \sim \triangle CDF$. Using those ratios: $\triangle AHE \sim \triangle CDF \sim \triangle BIF, \cdots$

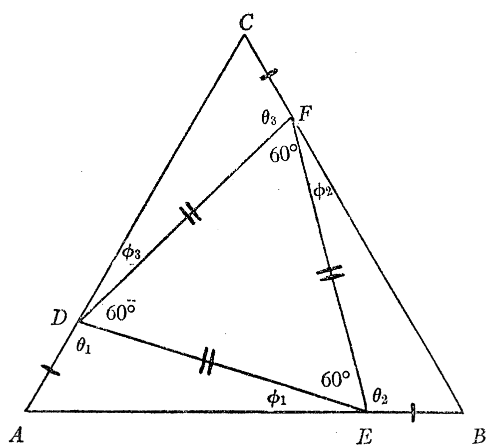
$$\therefore \triangle BFE \cong \triangle AED \cong \triangle CDF$$

$$\therefore m\angle 8 = m\angle 7 = m\angle 9 = 60^\circ \Leftrightarrow ABC \text{ is equilateral.}$$



III. *Solution by Michael Goldberg, Washington, D. C.*

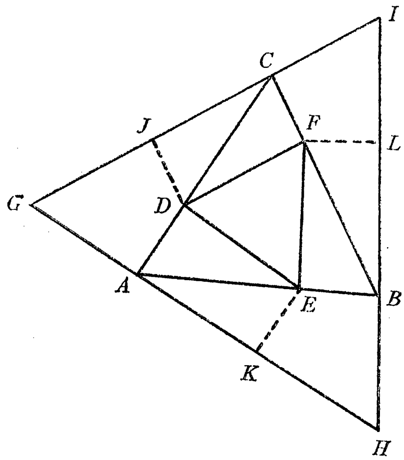
Let the largest angles at D, E, F be θ_1, θ_2, θ_3 respectively, and the smallest angles be ϕ_3, ϕ_1, ϕ_2. If $A > 60^\circ$, then $\theta_1 + \phi_1 < 120^\circ$, from which $\theta_1 < 120^\circ - \phi_1$. At point E, $\theta_2 = 180^\circ - 60^\circ - \phi_1 = 120^\circ - \phi_1 > \theta_1$. Therefore, $BF > AE$, and $\phi_2 < \phi_1$, $\theta_3 > \theta_2$, $\phi_3 < \phi_2$, $\theta_1 > \theta_3$, from which $\theta_1 < \theta_2 < \theta_3 < \theta_1$. This contradiction shows that angle A, or B or C, cannot be greater than 60°.



IV. *Solution by Frank J. Papp, University of Lethbridge, Canada.*

The following more general result will be demonstrated:

Triangle ABC is equilateral if and only if triangle DEF is equilateral (given that $\overline{AD}=\overline{BE}=\overline{CF}$).



1. Suppose that $\triangle ABC$ is equilateral and that $\overline{AD}=\overline{BE}=\overline{CF}$. Then $\overline{CD}=\overline{AE}=\overline{BF}$. From the law of cosines, we obtain

$$\begin{aligned}\overline{DF}^2 &= \overline{CD}^2 + \overline{CF}^2 - 2\,\overline{CD}\,\overline{CF}\cos C\\ &= \overline{AE}^2 + \overline{AD}^2 - 2\,\overline{AE}\,\overline{AD}\cos A\\ &= \overline{DE}^2. \quad \text{Thus } \overline{DF} = \overline{DE}.\end{aligned}$$

Similarly, $\overline{DF}=\overline{EF}$.

2. Construct $\triangle GHI$ with

$$\begin{aligned} A &\in \overline{GH} \quad \text{and} \quad \overline{GH}//\overline{DE}\\ B &\in \overline{HI} \quad \text{and} \quad \overline{HI}//\overline{EF}\\ C &\in \overline{GI} \quad \text{and} \quad \overline{GI}//\overline{DF}.\end{aligned}$$

Since $\triangle GHI$ and $\triangle DEF$ are similar and $\triangle DEF$ is equilateral, so is $\triangle GHI$.

Construct $\overline{DJ}$ parallel to $\overline{CF}$ intersecting $\overline{GI}$ at J.

Similarly construct $\overline{EK}$ and $\overline{FL}$.

Then, $ADEK$, $BEFL$, and $CFDJ$ are congruent parallelograms and thus corresponding diagonals are equal, in particular $\overline{AE}=\overline{CD}=\overline{EF}$.

From this and the fact $\overline{AD}=\overline{BE}=\overline{CF}$, we obtain $\overline{AB}=\overline{BC}=\overline{CD}$.

(3. *Remark*: In the degenerate cases $A=D$ or $\overline{AD}=\overline{AC}$ the results are immediate and there is nothing to prove. Thus there is no loss in generality in assuming the point D to be strictly between A and C.)

Also solved by Walter Blumberg, Flushing High School, New York; Haig Bohigian, John Jay College of Criminal Justice; Derrill J. Bordelon, Naval Underwater Weapons Research and Engineering Station, Newport, Rhode Island; Huseyin Demir, Middle East Technical University, Turkey; William F. Fox, Moberly, Missouri; M. G. Greening, University of New South Wales, Australia; Jeffrey Hoffstein, Bronx High School of Science, New York; Masatoshi Ikeda, Middle East Technical University, Turkey; V. F. Ivanoff, San Carlos, California; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Henrietta O. Midonick, New York, New York; Robin Milnor, University College of Swansea, United Kingdom; Prasert Na Nagara, Kasetsart University, Bangkok, Thailand; D. F. Paget, University of Tasmania; Sidney Penner, Bronx Community College, New York; P. G. Pantelidakis, Phoenix, Arizona; Wayne K. Peterson, Southern California College; Donald E. Rossi, DeAnza College, Cupertino, California; E. F. Schmeichel, College of Wooster, Ohio; Benjamin L. Schwartz, The Mitre Corporation, McLean, Virginia; Harry Sitomer, C. W. Post College, New York; William L. Veeck, Kauai Community College, Hawaii; Dick A. Wood, Seattle Pacific College, Washington; and Gregory Wulczyn, Bucknell University, Pennsylvania.

Several solvers pointed out the necessity for the points D, E and F to lie on the interior of the segments AB, BC and CA.

Monic Polynomial Property

755. [March, 1970] *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Let $f(x)=x^2+px+q$ where p and q are real numbers. Show that at least one of the numbers $|f(1)|$, $|f(2)|$, or $|f(3)|$ is not less than one-half. Generalize to a polynomial of degree n.

Solution by Steve Rohde, Lehigh University.

Since $f(x)$ is a monic polynomial of degree 2 we must have that the second forward differences are constant and equal to 2. Thus $f(3)-2f(2)+f(1)=2$ which is impossible if $|f(i)|<\frac{1}{2}$ for $i=1, 2, 3$. Likewise if f is a monic polynomial with real coefficients of degree n we must have

$$n! = \Delta^n f(1) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f(1+k) \leqq 2^n \max_{1\leqq i\leqq n+1} |f(i)|.$$

Hence at least one of $|f(i)|$, $1\leqq i\leqq n+1$, is not less than $n!/2^n$.

Also solved by Walter Blumberg, New Hyde Park, New York; Huseyin Demir, Middle East Technical University, Ankara, Turkey; M. G. Greening, University of New South Wales, Australia; Samuel P. Hoyle, Jr., College of William and Mary; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; Stefan Porubsky, Zvolen, Czechoslovakia; E. F. Schmeichel, College of Wooster, Ohio; Harry Sitomer, C. W. Post College, New York; G. P. Speck, Bradley University, Illinois; Gregory Wulczyn (partially), Bucknell University; Robert L. Young, Cape Cod Community College; and the proposer.

Centrally Symmetric Curves

756. [March, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine closed and centrally symmetric curves C, other than circles, such that the product of two perpendicular radius vectors (issued from the center) be a constant.

Solution by Harry W. Hickey, Arlington, Virginia.

Let us consider a generalized form of the problem: "Determine closed and centrally symmetric curves C, other than circles, such that the product of two radius vectors (issued from the center) be a constant, when the angle between the radius vectors is π/N, N being any positive even integer." We will call the center O, while the constant product is R^2. Construct a circle K of center O and radius R. Now for every point of C inside the circle, there is a point outside it, such that the product of the distances of the two points from O is R^2. Hence C crosses K at some point, say A, and crosses again at point B, where $\angle AOB=\pi/N$. Draw an arc from A to B which does not pass through O, nor intersect any line through O more than once—aside from these restrictions, the form of the arc is arbitrary. Let the polar equation of this arc be $\rho=f(\theta)$. The restrictions we have placed on the form of the arc insure that the reciprocal of f is always finite, and that f is single-valued—a multivalued f leads to ambiguities about the length of the radius vector. So far, f is defined for values of θ in a domain of length π/N. We can extend this to other values of θ by writing

$$f(\theta + \pi/N) = R^2/f(\theta) \quad \text{for all } \theta,$$

and the the curve C is defined! Because f is now periodic, of period $2\pi/N$, and since N is even, we have $f(\theta+\pi)=f(\theta)$, making C centrally symmetric (drop the symmetry requirement, and N can be odd).

Also solved by Michael Goldberg, Washington, D. C.; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; John Oman, Wisconsin State University, Oshkosh; E. F. Schmeichel, College of Wooster, Ohio; and the proposer.

A Divisibility Property

757. [March, 1970] *Proposed by Erwin Just, Bronx Community College, New York.*

Let p and q be primes and x an integer such that

$$p \,\Big|\, \sum_{k=0}^{q^r-1} x^k, \qquad (r \geqq 1).$$

Prove that either $p-1$ is divisible by q or $p=q$.

I. *Solution by Yul J. Inn, Aragon High School, San Mateo, California.*

Suppose $p\neq q$. Then $x\not\equiv 1 \pmod p$. This so because if $x\equiv 1 \pmod p$, then

$$0 \equiv \sum_{k=0}^{q^r-1} x^k \equiv q^r \pmod p,$$

which implies $p=q$. In particular, $x\neq 1$. Note also that $p\neq 2$ for if $p=2$ and $q\neq 2$, then $\sum_{k=0}^{q^r-1} x^k$ is odd. This is a contradiction.

Since $x\neq 1$, we have

$$p \,\Big|\, \sum_{k=0}^{q^r-1} x^k = \frac{x^{q^r}-1}{x-1}.$$

Thus we have

$$x^{q^r} \equiv 1 \pmod{p}. \tag{1}$$

This leads to two cases:

i) $q^r \leqq p-1$. Then $q^r | p-1 \Rightarrow q | p-1$.

ii) $q^r > p-1$. Let $q^r \equiv m \pmod{(p-1)}$ with $0 \leqq m < p-1$. If $m \equiv 0 \pmod{(p-1)}$, then $p-1 | q^r \Rightarrow q^s = p-1$, $s \geqq 1 \Rightarrow q | p-1$. $m \equiv 1 \pmod{(p-1)}$ implies $x \equiv 1 \pmod{p}$. Therefore let $1 < m < p-1$. Since $x^m \equiv 1 \pmod{p}$, $m | p-1$. Letting $p-1 = mn$ with $1 < n < p-1$, we have $nq^r \equiv 0 \pmod{(p-1)}$ and thus $q | p-1$.

II. *Solution by Mabel Szeto, Staten Island Community College, New York.*

Let $s = \sum_{k=0}^{q^r-1} x^k$. We have two cases:

Case 1: $x = 1$. Then $s = 1+1+ \cdots +1 = q^r$ (since the sum has q^r number of terms of ones). So the given condition that $p | s$ becomes $p | q^r$; but p and q are both primes, hence we must have $p = q$.

Case 2: $x \neq 2$. Then $s = (x^{q^r}-1)/(x-1)$. Since it is given that $p | s$, there exists an integer n such that $x^{q^r}-1 = p(x-1)n$. This equality shows that $p \nmid x$ otherwise p will have to divide into the 1 on the left of the relation, which can be restated as $x^{q^r} \equiv 1 \pmod{p}$. But since $(x, p) = 1$, by Fermat's theorem, we also have $x^{(p-1)} \equiv 1 \pmod{p}$. Now let m be the smallest integer such that $x^m \equiv 1 \pmod{p}$, then we must have $m | q^r$ and $m | p-1$. But $m | q^r$ implies $q^r = m \cdot t = q^l \cdot q^{r-l}$, $1 \leqq l \leqq r$ (since q is prime, this is the only decomposition for q^r), and $m | p-1$ implies $p-1 = m \cdot s = q^l \cdot s$, which in turn shows that $q | p-1$.

Also solved by Walter Blumberg, Flushing High School, New York; M. G. Greening, University of New South Wales, Australia; Dean Hickerson, Davis, California; E. F. Schmeichel, College of Wooster, Ohio; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Wilke pointed out that the problem is a generalization of Problem 19.10 appearing on Page 132 of *Theory of Numbers*, by B. M. Stewart, second edition.

An Angle Bisector

758. [March, 1970] *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

If the internal bisector of angle A of a triangle ABC is perpendicular to the line joining the incenter and the orthocenter, show that $\cos B + \cos C = 1$.

Solution by Leon Bankoff, Los Angeles, California.

Let D denote the foot of the altitude from A upon BC and let E denote the projection of the incenter I upon AH, where H is the orthocenter. Then

$$\begin{aligned} IH^2 &= AH \cdot EH \\ &= AH(r - HD) \\ &= 2Rr \cos A - 4R^2 \cos A \cos B \cos C. \end{aligned}$$

In any triangle, it is known that

$$IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$$

Hence $2Rr\cos A = 2r^2$, and $\cos A = r/R$.

Since $\cos A + \cos B + \cos C = 1 + r/R$, it follows that $\cos B + \cos C = 1$.

Additional items of interest in this configuration:

[1] OI is parallel to BC.

[2] $AH = 2r$.

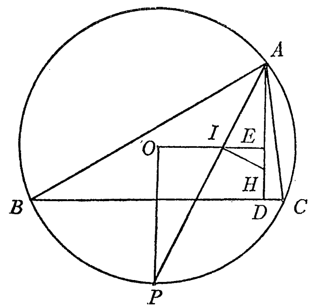
[3] $AI/AP = \cos A$, where P is the point of intersection of AI and the circumcircle of triangle ABC.

[4] The circumcircle of triangle AIH is equal to the incircle of triangle ABC.

[5] $IP \cdot AI = 2Rr = \sqrt{R \cdot AI \cdot BI \cdot CI}$.

[6] $\sin^2(B/2) + \sin^2(C/2) = 1/2$; $\cos^2(B/2) + \cos^2(C/2) = 3/2$.

[7] $\tan^2(A/2) = (R - r)/(R + r)$.



Also solved by Huseyin Demir, Middle East Technical University, Ankara, Turkey; M. G. Greening, University of New South Wales, Australia; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; E. F. Schmeichel, College of Wooster, Ohio; Harry Sitomer, C. W. Post College, New York; A. W. Walker, Toronto, Canada (two solutions); and the proposer.

The Problem of Apollonius

759.* [March, 1970] *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Circles A, C, and B with radii of lengths a, c, and b, respectively, are in a row, each tangent to a straight line DE. Circle C is tangent to circles A and B. A fourth circle is tangent to each of these three circles. Find the radius of the fourth circle.

Solution by Derrill J. Bordelon, Naval Underwater Weapons Research and Engineering Station, Newport, Rhode Island.

The stated problem is a special case of the problem of Apollonius [*The Problem of Apollonius*, The Mathematics Teacher, N. A. Court, Vol. 54, No. 6, Oct. 1961, pp. 444–452]. Solutions of Apollonius' problem by simple algebraic methods and by inversion [*What is Mathematics?*, R. Courant and H. Robbins, Oxford Univ. Press, 1941] are well known. In the most general case if the number of solutions is finite, then there are at most eight. From the symmetric determinant [*A Treatise on Conic Sections*, George Salmon, sixth ed., Chelsea, New York, p. 134] connecting the mutual distances of any four points in the plane, where R is the radius of the oriented tangent circle, one finds:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \{(b-a)^2+4(\sqrt{ac}+\sqrt{bc})^2\} & (a+c)^2 & (R-e_1a)^2 \\ 1 & \{(b-a)^2+4(\sqrt{ac}+\sqrt{bc})^2\} & 0 & (c+b)^2 & (R-e_2b)^2 \\ 1 & (a+c)^2 & (c+b)^2 & 0 & (R-e_3c)^2 \\ 1 & (R-e_1a)^2 & (R-e_2b)^2 & (R-e_3c)^2 & 0 \end{vmatrix} = 0$$

where $e_i = \pm 1$, $i = 1, 2, 3$. Thus, there are 8 determinants and 8 cases to be considered for each of the rows of the following matrix:

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \\ +1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & +1 \\ +1 & +1 & +1 \end{bmatrix}$$

where

$$E_j = [e_1 e_2 e_3], \qquad j = 1, 2, \cdots, 8.$$

Thus, $e_i = \left\{\begin{smallmatrix} -1 \\ +1 \end{smallmatrix}\right\}$ indicates that the ith given circle of positive radius r_i, viz., $r_1 = a$, $r_2 = b$, $r_3 = c$ and the desired tangent oriented circle of radius R_j are tangent $\left\{\begin{smallmatrix} \text{externally} \\ \text{internally} \end{smallmatrix}\right\}$ when R_j is positive, $j = 1, 2, \cdots, 8$. When R_j is negative, $e_i = \left\{\begin{smallmatrix} -1 \\ +1 \end{smallmatrix}\right\}$ corresponds to the case where they are tangent $\left\{\begin{smallmatrix} \text{internally} \\ \text{externally} \end{smallmatrix}\right\}$.

Accordingly, R_j, the radius of the tangent circle in the jth case, i.e., for E_j, $j = 1, 2, \cdots, 8$, is as follows:

$$R_1 = \begin{Bmatrix} \infty \text{ (the given tangent line } DE) \\ \dfrac{c}{4}\left[\dfrac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{2}{\sqrt{ab}}}{\dfrac{1}{c}-\dfrac{1}{\sqrt{ab}}}\right] \end{Bmatrix}$$

$$R_2 = c\left[\dfrac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{2}{\sqrt{ab}}}{\dfrac{1}{c}-\dfrac{1}{b}-\dfrac{2}{\sqrt{ab}}}\right],$$

$$R_3 = c\left[\dfrac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{2}{\sqrt{ab}}}{\dfrac{1}{c}-\dfrac{1}{a}-\dfrac{2}{\sqrt{ab}}}\right],$$

$$R_4 = c, \quad R_5 = -R_4, \quad R_6 = -R_3, \quad R_7 = -R_2, \quad R_8 = -R_1.$$

Also solved by Merrill Barnebey, Wisconsin State University, La Crosse; and Huseyin Demir, Middle East Technical University, Ankara, Turkey.

Cyclotomic Fields

760. [March, 1970] *Proposed by Hugh M. Edgar, Chris W. Avery, and Albert E. Pollatchek, San Jose State College, California.*

Characterize those positive integral values of n and m for which the nth cyclotomic field is contained in the mth cyclotomic field.

Solution by the proposers.

Let ζ_n denote any primitive nth root of unity, Q the field of rational numbers so that $Q(\zeta_n)$ denotes the nth cyclotomic field. If $\zeta_n \in Q(\zeta_m)$ then ζ_n is an element of order n in the infinite abelian group $(Q(\zeta_m))^* = Q(\zeta_m) - \{0\}$. Likewise ζ_m is an element of order m in this same group and hence there necessarily exists an element α in the group $(Q(\zeta_m))^*$ having order $[n, m]$ = the least common multiple of m and n (see, for instance, Lemma 1, p. 49, of [1]). Since α is an algebraic number of degree $\phi([n, m])$ (ϕ denoting the Euler ϕ-function) and since $Q(\alpha) \subseteq (\zeta_m)$ we require $\phi([n, m]) \mid \phi(m)$. However $m \mid [n, m]$ from which it follows, from elementary number theory, (see, for instance, #3, p. 31, of *Topics in Number Theory*, by W. J. LeVeque, Addison-Wesley—Theorem 3–8, p. 29 can be used to supply a proof) that $\phi(m) \mid \phi([n, m])$. Thus it has been established that $Q(\zeta_n) \subseteq Q(\zeta_m)$ requires $\phi([n, m]) = \phi(m)$. The solution set of the equation $\phi([n, m]) = \phi(m)$ is easily determined: We must either have $n \mid m$ or $n = 2d$ where $d \mid m$ and m is odd.

If $n \mid m$ it follows that $Q(\zeta_n) \subseteq Q(\zeta_m)$ since $Q(\zeta_m)$ contains all mth roots of unity and ζ_n is an mth root of unity. Assume now that $n = 2d$ where $d \mid m$ and m is odd. ζ_d is an element of order d in the infinite abelian group $(Q(\zeta_n))^* = (Q(\zeta_{2d}))^*$ while -1 is an element of order 2 in the same group. Since $(2, d) = 1$ it follows that $-\zeta_d$ is an element of order $2d$ in $(Q(\zeta_n))^*$.

Hence $Q(-\zeta_d) = Q(\zeta_d) = Q(\zeta_n)$. However $d \mid m$ and so $Q(\zeta_n) \subseteq Q(\zeta_m)$. Thus it has been established that the condition $\phi([n, m]) = \phi(m)$ implies $Q(\zeta_n) \subseteq Q(\zeta_m)$. Hence $Q(\zeta_n) \subseteq Q(\zeta_m)$ if and only if $\phi([n, m]) = \phi(m)$.

Reference

1. Emil Artin, Galois Theory, Notre Dame, Indiana, 2nd ed., 1959.

Comment on Problem 735

735. [September, 1969, and March, 1970] *Proposed by Charles W. Trigg, San Diego, California.*

Find a triangular number which can be partitioned into three 3-digit primes which together contain the nine positive digits.

Comment by the proposer.

There is no need to resort to a computer. This problem can be solved expeditiously, possibly in less time than it took to write the computer program.

3-digit primes must terminate in 1, 3, 7 or 9, which also are the only possible terminations of the sum, T, of three such primes with distinct digits. Since all nine positive digits appear in the three primes, $T \equiv 0 \pmod 9$. Without further consideration of primacy,

$$153 + 267 + 489 = 909 \leqq S \leqq 2451 = 957 + 843 + 621.$$

Triangular numbers, $T = n(n+1)/2$, within this range and with the proper terminations are 1081, 1431, 1653, 1711, 1891, 1953 and 2211. Only 1431 and 1953 are $\equiv 0 \pmod 9$.

For $T = 1431$, the units' digits of the primes must be 1, 3 and 7. Then the sum of the tens' digits is $12 = 2+4+6$, with the sum of the remaining digits being >13; or, the tens' digits are 5, 8, 9 and the hundreds' digits are 2, 4, 6. These digits can be distributed in (6)(6) ways. The possibilities are further reduced since the only primes beginning with 6 that fit the pattern are 653, 683, 691. Thus the two solutions are found:

$$1431 = 491 + 683 + 257 = 691 + 283 + 457.$$

Similarly, for $T = 1953$, the units' digits are 1, 3, 9. Then the tens' and hundreds' digits are 2, 4, 8 and 5, 6, 7, respectively; or 2, 5, 7 and 4, 6, 8. The resulting two sets of 3-digit primes are

$$1953 = 421 + 673 + 859 = 821 + 653 + 479.$$

Comment on Problem 736

736. [September, 1969, and March, 1970] *Proposed by Alfred Kohler, Long Island University.*

Al, Bill, Chuck, and Don all live in the same school district. When Al faces the school from his home, Bill's home is directly to Al's right. Bill lives directly to the west of school, and Chuck lives due south of Bill. At his home, Don can see the sun setting behind Chuck's house at times.

Al's home is as far from Bill's as it is from school, while Chuck lives twice as far from Bill as Bill does from school. Don lives three times as far from Chuck as Chuck does from school, and Don lives ten times as far from school as does Al.

Show that Al's home, Don's home, and the school are collinear.

Comment by Charles W. Trigg, San Diego, California.

There is a simple smooth synthetic solution that does not involve the arctan function.

In the figure on page 107, each building is represented by the initial letter of the inhabitant's name. AS is taken as 1. Then, progressively, $AB = 1$, $BS = \sqrt{2}$, $BC = 2\sqrt{2}$, $SC = \sqrt{10}$, $CD = 3\sqrt{10}$, and $SD = 10$, so SCD is a right angle. Cos $\alpha = 1/\sqrt{2}$, $\cos \beta = 1/\sqrt{5}$, and $\cos \gamma = 1/\sqrt{10}$. Then

$$\cos(\alpha+\beta) = (1/\sqrt{2})(1/\sqrt{5}) - (1/\sqrt{2})(2/\sqrt{5}) = -1/\sqrt{10}.$$

Whereupon,

$$\cos(\alpha+\beta+\gamma) = (-1/\sqrt{10})(1/\sqrt{10}) - (3/\sqrt{10})(3/\sqrt{10}) = -1$$

so angle $ASD = 180°$.

Comment on Q466

Q466. [November, 1969] If AB and BA are both identity matrices, then A and B are both square matrices.

[*Submitted by Murray S. Klamkin*]

Comment by J. L. Brenner, University of Arizona.

An alternative proof is the following. We first note the equations

$$\begin{bmatrix} \lambda I & A \\ 0 & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & -A \\ -B & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & 0 \\ B & \lambda I \end{bmatrix} = \begin{bmatrix} \lambda^2 I - AB & 0 \\ 0 & \lambda^3 I \end{bmatrix}$$

$$\begin{bmatrix} \lambda I & 0 \\ B & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & -A \\ -B & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & A \\ 0 & \lambda I \end{bmatrix} = \begin{bmatrix} \lambda^3 I & 0 \\ 0 & \lambda^2 I - BA \end{bmatrix}.$$

If $AB = I_r$, $BA = I_s$, then using $\det(XYZ) = \det X \det Y \det Z = \det(ZXY)$, $(\lambda^2-1)^r(\lambda^3)^s = (\lambda^3)^r(\lambda^2-1)^s$.

This cannot be valid for all λ except if $r = s$, which is what was to be shown. The only thing this proof uses is $\det(XY) = \det X \det Y = \det Y \det X$. The proof is therefore slightly more powerful than the one given on page 244.

Comment on Q472

Q472. [March, 1970] Let $\alpha \geqq 0$, $\beta \geqq 0$, and $\alpha+\beta \leqq \pi/2$. Prove

$$2 + \tan\alpha + \tan\beta \leqq 2\sec\alpha + 2\sec\beta.$$

[*Submitted by Harley Flanders*]

Comment by Samir K. Kar, Indiana University.

A simpler solution and one valid in a more general setting is the following:

$$\begin{aligned} 2 + \tan\alpha + \tan\beta &= (1+\tan\alpha) + (1+\tan\beta) \\ &= \left(\frac{\cos\alpha + \sin\alpha}{\cos\alpha}\right) + \left(\frac{\cos\beta + \sin\beta}{\cos\beta}\right) \\ &\leqq \left(\frac{1+1}{\cos\alpha}\right) + \left(\frac{1+1}{\cos\beta}\right) \\ &= 2\sec\alpha + 2\sec\beta. \end{aligned}$$

Comment on Q476

Q476. [March, 1970] Show that

$$z = \sqrt{z\sqrt{z\sqrt{z\sqrt{z}}}} \cdots$$

[*Submitted by P. G. Pantelidakis*]

Comment by Paul A. Oskay, Downey, California.

Another simple proof follows:

$$z = z^{1/2} \cdot z^{1/4} \cdot z^{1/8} \cdots = z^{1/2+1/4+1/8+\cdots}$$
$$= z \exp \sum_{n=1}^{\infty} 1/2^n$$

but

$$\sum_{n=1}^{\infty} 1/2^n = 1 + \sum_{n=1}^{\infty} 1/2^n - 1 = \sum_{n=0}^{\infty} 1/2n - 1 = 1/(1 - 1/2) - 1 = 1.$$

Therefore $z = z$.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q487. Under what circumstances can $1010 = 10$?

[*Submitted by Charles W. Trigg*]

Q488. Why is it that every multiplicative numbertheoretic function always has the value 1 at 1?

[*Submitted by John D. Baum*]

Q489. Suppose that $\int_a^b f(x)dx = 0$. Then $\int_a^b (x-c)f(x)dx$ is independent of c. If we choose c to be the x-coordinate of the centroid of the area under the curve $y = f(x)$ then $\int_a^b (x-c)f(x)dx = 0$. Whence, $\int_a^b xf(x)dx = 0$. Explain.

[*Submitted by Ralph Boas*]

Q490. If x is any integer, prove that

$$x^4 + 2x^3 + 5x^2 + 4x + 5$$

is not divisible by 67.

[*Submitted by Erwin Just*]

Q491. Suppose the length of the hypotenuse of a right triangle whose legs have rational lengths is a prime. Prove that the altitude of the triangle erected on the hypotenuse is not an integer.

[*Submitted by A. A. Mullin*]

(Answers on page 249.)

eral problem for $r \cdot 3^s$, where $s=1, 2, 3, \cdots$ and $r=7, 9, 13, 15, 19, 31, 33, 69, 75, 87, 111, 123$, or 127.

The methods given in this paper exhibit the existence of several infinite classes of $BL(k, n)$ spaces. We note however, that in each case $k \geqq n$ for the spaces that we have found. A natural question is, "Do finite $BL(k, n)$ spaces exist for $n > k$? In particular, does the smallest possible $BL(k, n)$ space ($n > k$), namely $BL(3, 6)$ exist? Also, is there a $BL(k, 3)$ space for each integer k, not ruled out by Theorem 2?" or, "Are there any values of k and n for which $BL(k, n)$ does not exist excepting for those ruled out by Theorem 2?" There is also the question of the existence of other rectangular systems, hence the possible existence of other $BL(k, n)$ spaces, which needs to be answered more fully.

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2. E. H. Moore, Concerning the general equations of the seventh and eighth degrees, Math. Ann., 51 (1898) 417–444.

3. W. W. R. Ball, Mathematical Recreations and Essays, rev. by H. S. M. Coxeter, Macmillan, London, 1940.

4. Oswald Veblen and W. H. Bussey, Finite projective geometries, Trans. Amer. Math. Soc., 7 (1906) 241–259.

ANSWERS

A487. $1010_a = 10_b$ when $a + a^3 = b$. For example: when $a=2, b=10$ or $a=3, b=30$. Otherwise, if multiplication dots are placed between the last 0 and the preceding 1 on each side of the equality sign, $0=0$ results.

A488. Otherwise it would be identically zero, for since $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$ we must have $f(1) = 1$ or 0. If $f(1) \neq 1$, then it must satisfy $f(1) = 0$ and $f(m) = f(1) \cdot f(m) = 0$.

A489. Since $\int_a^b f(x)dx = 0$, there need not exist a centroid. For the discreet case, just consider a couple (two equal and opposite forces).

A490. It is known that the only odd prime divisors of N^2+1 are of the form $4n+1$. Since 67 is a prime which is not of the form $4n+1$ and $x^4+2x^3+5x^2+4x+5 = (x^2+x+2)^2+1$ the conclusion follows.

A491. Let c be the length of the hypotenuse and h be the length of the altitude erected on the hypotenuse. Consider the area of the triangle

$$A = ab/2 = hc/2; \quad \text{thus } h = ab/c.$$

Suppose h is an integer. Since c is prime, either $c|a$ or $c|b$. This is impossible since $a < c$ and $b < c$ as the length of the hypotenuse always exceeds the length of either leg. The result follows from the contradiction.

(Quickies on page 291.)

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